2 Adverse Selection

Information asymmetries concerning types
Basic model

- There is a principal (e.g., a firm) and an agent (e.g., a supplier)
- The agent delivers the quantity $q$ of a good to the principal
- The principal pays the transfer $t$ to the agent

Graphically:
Principal’s utility

- The principal’s benefit of receiving quantity $q$ is $s(q)$, where $s’ > 0$, $s'' < 0$, $s(0) = 0$, $s’(0) = \infty$
  - Example 1: $s(q) = \sqrt{q}$
  - Example 2: Principal’s production technology is $f(k, q) = k^{1-\alpha} q^\alpha$, with $\alpha \in (0, 1)$, and the price of an output unit is $p \rightarrow$ her revenue/benefit is $pk^{1-\alpha} q^\alpha$

- Principal’s utility is
  
  $$u_P = s(q) - t$$
Agent’s utility

Agent’s utility is

\[ u_A = t - c(q, \theta), \]

where \( c(q, \theta) \) are the agent’s production costs

The agent’s costs depend on his type \( \theta \), which is either low \( \underline{\theta} \) or high \( \bar{\theta} \), with \( 0 < \theta < \bar{\theta} \)

For concreteness, we suppose that

\[ c(q, \theta) = \theta q \]
The agent’s type $\theta$ is drawn and learned by the agent, where $\phi := \text{prob}(\theta = \overline{\theta})$ and $\phi \in (0, 1)$

The principal suggests a contract $(q, t)$ or a menu (i.e., set) of contracts $\{(q, t)\}$

The agent either accepts or rejects

- If the agent rejects, the game ends and parties receive reservation utilities of $u^\text{res}_P = u^\text{res}_A = 0$
- If the agent accepts, the game continues and he chooses a contract if there is a menu of contracts

The agent produces and delivers the contracted quantity to the principal; the principal pays the agent the contracted transfer
Benchmarks

Before we analyze the previously described model, we will consider two benchmarks that are useful to have comparisons:

1. Benchmark I: welfare maximum, where a social planner maximizes welfare
2. Benchmark II: symmetric information, where also the principal knows the agent’s type
Benchmark I: welfare maximum

• Suppose there is a social planner, who seeks to maximize welfare
• The planner knows the agent’s type and can enforce the desired quantities
• Welfare is \( V = u_P + u_A = s(q) - t + t - c(q, \theta) = s(q) - \theta q \)
• \( s(q) - \theta q \) can be interpreted as the surplus of the transaction
• Maximizing over \( q \) (solving \( \partial V / \partial q = 0 \)) yields that the efficient – i.e., welfare maximizing – quantity \( q^{effi} \) solves \( s'(q) = \theta \)
• The efficient quantity is thus obtained by equating the principal’s marginal benefit and the agent’s marginal costs
• The efficient quantity is type dependent:
  • If the agent is of type \( \theta \), the efficient quantity \( q^{effi} \) solves \( s'(q) = \theta \)
  • If the agent is of type \( \bar{\theta} \), the efficient quantity \( \bar{q}^{effi} \) solves \( s'(q) = \bar{\theta} \)
• While the transfer influences each parties’ utility, it does not influence welfare \( \implies \) all transfers are efficient
It holds that $q_{effi} > \bar{q}_{effi} > 0$

Proof:

- We know that $q_{effi}$ is determined by $s'(q) = \theta$
- For $\theta$, the right-hand side is lower than for $\bar{\theta}$ due to $\theta < \bar{\theta}$
- Hence, for $\theta$, also the left-hand side must be lower than for $\bar{\theta}$
- Because $s$ is concave (i.e., $s'(q)$ is decreasing in $q$) it has to hold that $q_{effi} > \bar{q}_{effi}$
- Moreover, $\bar{q}_{effi} > 0$ must hold due to $s'(0) = \infty$

Alternative proof by graphic

Intuition: If the agent has lower production costs (i.e., if he is of type $\theta$ and not of type $\bar{\theta}$), it is efficient to produce more
### Proposition 1

If the agent is of type $\theta$, the efficient quantity $q^{effi}$ solves $s'(q) = \theta$. If the agent is of type $\bar{\theta}$, the efficient quantity $\bar{q}^{effi}$ solves $s'(q) = \bar{\theta}$. It holds that $q^{effi} > \bar{q}^{effi} > 0$.

**Example:**

- Suppose $s(q) = \sqrt{q}$, $\theta = 1$, and $\bar{\theta} = 2$
- In the welfare maximum, we must have that $s'(q) = \frac{1}{2} q^{-1/2}$ and so $q^{effi} = \frac{1}{4\theta^2}$
- Hence, $q^{effi} = \frac{1}{4}$ and $\bar{q}^{effi} = \frac{1}{16}$
Benchmark II: symmetric information

- Suppose that the agent and the principal know the agent’s type (symmetric information)
- We call this scenario the first-best
- The principal maximizes her utility $u_P$ over $q$ and $t$, subject to the constraint that the agent accepts her offer
- The agent accepts, i.e., participates, if this is at least weakly better than not accepting:
  \[ t - \theta q \geq 0 \] (PC)
- This constraint is called participation constraint (in some books individual rationality constraint)
Optimum

- The Lagrangian writes
  \[ \mathcal{L}(q, t) = s(q) - t + \lambda(t - \theta q) \]

- In the optimum, the following holds (cf. the “Simple Recipes for Optimization”):
  \[ \frac{\partial \mathcal{L}(\cdot)}{\partial t} = -1 + \lambda = 0 \quad (1) \]
  \[ \frac{\partial \mathcal{L}(\cdot)}{\partial q} = s'(q) - \lambda \theta = 0 \quad (2) \]
  \[ t - \theta q \geq 0 \quad \text{(PC)} \]
  \[ \lambda \geq 0; \quad \text{if } t - \theta q > 0, \text{ then } \lambda = 0 \quad (3) \]

- Remark: The complementary slackness condition “if \( t - \theta q > 0 \), then \( \lambda = 0 \)” can also be written as \( \lambda(t - \theta q) = 0 \)
Solving the system

- From (1) we get that
  \[ \lambda^{FB} = 1 \]

- Since \( \lambda^{FB} = 1 \), the \((PC)\) must bind (see the “Simple Recipes for Optimization”\(^1\)) and so
  \[ t^{FB} = \theta q^{FB} \]

- Intuition: the principal optimally chooses the transfer as low as possible, i.e., such that the participation constraint holds with equality

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\(^1\)This can also be seen directly from the system: since \( \lambda^{FB} = 1 \), by (3), \( t - \theta q > 0 \) cannot hold and so by \((PC)\), we must have \( t - \theta q = 0 \)
Plugging $\lambda^{FB} = 1$ into (2) yields that the principal optimally chooses the quantity $q^{FB}$ such that

$$s'(q) = \theta$$

If $\theta = \underline{\theta}$, $q^{FB}$ solves $s'(q) = \underline{\theta}$; if $\theta = \bar{\theta}$, $\bar{q}^{FB}$ solves $s'(q) = \bar{\theta}$

These are the efficient quantities: $q^{FB} = q^{effi}$ and $\bar{q}^{FB} = \bar{q}^{effi}$!

Intuition: If the parties would not maximize the surplus of the transaction, i.e., would not choose the efficient quantity, they could improve

This is a general insight (Coase Theorem): Without information problems or other transaction costs, parties bargaining to an efficient solution
Proposition 2

With symmetric information, the principal optimally sets the contract
\((q^{FB} = q^{effi}, t^{FB} = \theta q^{FB})\) if the agent is of type \(\theta\) and the contract
\((\bar{q}^{FB} = \bar{q}^{effi}, \bar{t}^{FB} = \bar{\theta} \bar{q}^{FB})\) if the agent is of type \(\bar{\theta}\).

Example:

- Suppose \(s(q) = \sqrt{q}, \theta = 1\), and \(\bar{\theta} = 2\)
- The principal optimally sets the contract \((q^{FB} = \frac{1}{4}, t^{FB} = \frac{1}{4})\) if the agent is of type \(\theta\) and the contract \((\bar{q}^{FB} = \frac{1}{16}, \bar{t}^{FB} = \frac{1}{8})\) if the agent is of type \(\bar{\theta}\)
Main scenario: asymmetric information

- Only the agent knows his type (asymmetric information)
- Recall that the agent can be of two types, \( \theta \) or \( \bar{\theta} \)
- Suggesting more than two contracts makes no sense, since then some contract(s) will never be chosen by any type of the agent
- Because contracts can be the same, it is at least weakly optimal to design two contracts: if we start with two contracts and optimize, it may turn out that both contracts should be the same
- We thus let the principal suggest two contracts:
  - one contract designated to type \( \theta \) denoted \( (q, t) \) and
  - another contract designated to type \( \bar{\theta} \) denoted \( (\bar{q}, \bar{t}) \)
Incentive constraints

- For each type the agent possibly has it must at least be weakly optimal for him to choose the designated contract.

- An agent of type $\theta$ must at least weakly prefer to choose contract $(q, t)$ over $(\bar{q}, \bar{t})$:

  $$t - \theta q \geq \bar{t} - \theta \bar{q} \quad (IC)$$

- An agent of type $\bar{\theta}$ must at least weakly prefer to choose contract $(\bar{q}, \bar{t})$ over $(q, t)$:

  $$\bar{t} - \bar{\theta} \bar{q} \geq t - \bar{\theta} q \quad (\bar{IC})$$

- Example: if $\theta = 1$ and $\bar{\theta} = 2$; the contracts $(q = 4, t = 6)$ and $(\bar{q} = 2, \bar{t} = 5)$ satisfy $(\bar{IC})$, but violate $(IC)$.
Implement the same contracts as in case of symmetric information?

- Is it possible to implement the same contracts as in case of symmetric information?
- No, since they are not incentive compatible
- To be precise, with the menu of contracts \( \{(q^{FB}, t^{FB}), (\bar{q}^{FB}, \bar{t}^{FB})\} \) the (IC) is violated
  - If the agent is of type \( \theta \), he yields utility \( u_A = 0 \) if choosing the contract \( (q^{FB}, t^{FB}) \), since the transfer then just compensates him for his costs
  - ... but a positive utility \( u_A > 0 \) if choosing the contract \( (\bar{q}^{FB}, \bar{t}^{FB}) \), since the transfer is then higher as his costs: \( \bar{t}^{FB} = \bar{\theta} \bar{q}^{FB} > \theta \bar{q}^{FB} \)
Implementable quantities

- What quantities $q$ and $\bar{q}$ can be implemented?
- Adding both incentive constraints yields
  \[
  (\bar{\theta} - \theta) (q - \bar{q}) \geq 0
  \]
- Since $\bar{\theta} > \theta$ it has to hold that $q \geq \bar{q}$
- Thus, no matter what contracts the principal designs, she can never design contracts for which $q < \bar{q}$
- Idea:
  - The low-cost type $\theta$ has a cost advantage over the high-cost type $\bar{\theta}$, which is higher for higher quantities
  - Accordingly, if the high-cost type should produce more than the low-cost type ($\bar{q} > q$) and prefers the contract $(\bar{q}, \bar{t})$, then also the low-cost type prefers the contract $(\bar{q}, \bar{t})$
Participation constraints

- If not explicitly stated differently, we suppose throughout the course that the principal always wants to make sure that the agent accepts a contract, no matter what type he has.

- The agent accepts if he receives a utility which is at least as large as his reservation utility.

- For an agent of type $\theta$ it thus has to hold that
  \[ t - \theta q \geq 0 \]  
  \((PC)\)

- For an agent of type $\bar{\theta}$ it thus has to hold that
  \[ \bar{t} - \bar{\theta} \bar{q} \geq 0 \]  
  \((\overline{PC})\)

- Example: if $\theta = 1$ and $\bar{\theta} = 2$, the contracts $(q = 4, t = 6)$ and $(\bar{q} = 2, \bar{t} = 3)$ satisfy $(PC)$, but violate $(\overline{PC})$.
Principal's problem

- The principal designs the menu of contracts to maximize her expected utility subject to the incentive and participation constraints.
- Formally, the principal

\[
\max_{\{(q,t), (\bar{q}, \bar{t})\}} E[u_P] \text{ subject to } (IC), (\bar{IC}), (PC), (\bar{PC})
\]
Neglect two constraints

- The participation constraint \((PC)\) holds automatically:

\[
\begin{align*}
\bar{t} - \theta \bar{q} & \geq \bar{t} - \theta \bar{q} \\
\bar{t} - \theta \bar{q} & \geq \bar{t} - \bar{\theta} \bar{q} & \geq 0 \\
\Rightarrow \quad t - \theta q & \geq 0
\end{align*}
\]

- We can hence neglect \((PC)\)

- We suppose that if the agent is of the high-cost type \(\bar{\theta}\), he does not want to mimic the low-cost type \(\theta\)

- We are hence confident that the incentive constraint \((IC)\) does not bind and therefore neglect it

- Later, we will verify that \((IC)\) is indeed not binding for the optimal contracts
Principal’s relaxed problem

- The principal’s relaxed problem is hence
  \[
  \max_{\{(q, t), (\bar{q}, \bar{t})\}} \quad E[u_P] \text{ subject to } (IC), (\overline{PC})
  \]

- The Lagrangian writes
  \[
  L(q, t, \bar{q}, \bar{t}) = \phi (s(q) - t) + (1 - \phi) (s(\bar{q}) - \bar{t}) \\
  + \bar{\lambda} (\bar{t} - \theta \bar{q}) \\
  + \mu (t - \theta q - \bar{t} + \theta \bar{q})
  \]
Optimum

- In the optimum, the following holds:
  \[
  \frac{\partial L(\cdot)}{\partial t} = -\phi + \mu = 0 \quad (4)
  \]
  \[
  \frac{\partial L(\cdot)}{\partial \bar{t}} = -(1 - \phi) + \bar{\lambda} - \mu = 0 \quad (5)
  \]
  \[
  \frac{\partial L(\cdot)}{\partial q} = \phi s'(\bar{q}) - \mu \theta = 0 \quad (6)
  \]
  \[
  \frac{\partial L(\cdot)}{\partial \bar{q}} = (1 - \phi)s'(\bar{q}) - \bar{\lambda}\bar{\theta} + \mu \theta = 0 \quad (7)
  \]
  \[
  \bar{t} - \bar{\theta}\bar{q} \geq 0 \quad (\text{PC})
  \]
  \[
  t - \theta q \geq \bar{t} - \theta\bar{q} \quad (\text{IC})
  \]
  \[
  \bar{\lambda} \geq 0; \quad \text{if } \bar{t} - \bar{\theta}\bar{q} > 0, \text{ then } \bar{\lambda} = 0 \quad (8)
  \]
  \[
  \underline{\mu} \geq 0; \quad \text{if } \underline{t} - \underline{\theta}q > \bar{t} - \theta\bar{q}, \text{ then } \underline{\mu} = 0 \quad (9)
  \]
Remark

- The complementary slackness condition “if $\bar{t} - \bar{\theta} \bar{q} > 0$, then $\bar{\lambda} = 0$” can also be written as $\bar{\lambda}(\bar{t} - \bar{\theta} \bar{q}) = 0$

- The complementary slackness condition “if $\bar{t} - \bar{\theta} \bar{q} > \bar{t} - \bar{\theta} \bar{q}$, then $\mu = 0$” can also be written as $\mu(\bar{t} - \bar{\theta} \bar{q} - \bar{t} + \bar{\theta} \bar{q}) = 0$
Solving the system

- From (4) we directly get that
  \[ \mu^* = \phi \]

- Since \( \mu^* = \phi > 0 \), (IC) must bind (see the “Simple Recipes for Optimization”\(^2\)) and so
  \[ t^* = \theta q^* + \bar{t}^* - \theta \bar{q}^* \]

- Plugging \( \mu^* = \phi \) into (5) yields
  \[ \bar{\lambda}^* = 1 \]

- Since \( \bar{\lambda}^* > 0 \), (PC) must bind and so
  \[ \bar{t}^* = \bar{\theta} \bar{q}^* \]

\(^2\)This can also be seen directly from the system: since \( \underline{\mu}^* > 0 \), (9) implies that (IC) cannot hold with \( > \) and thus must hold with \( = \)
• Plugging $\mu^* = \phi$ into (6) yields
  \[ s'(\bar{q}^*) = \theta \]
• Hence, $\bar{q}^* = q^{FB}$
• Plugging $\mu^* = \phi$ and $\bar{\lambda}^* = 1$ into (7) yields
  \[ (1 - \phi)s'(\bar{q}) - \bar{\theta} + \phi\theta = 0 \]
• Rewriting this we obtain
  \[ s'(\bar{q}^*) = \frac{\bar{\theta} - \phi\theta}{1 - \phi} = \frac{(1 - \phi + \phi)\bar{\theta} - \phi\theta}{1 - \phi} = \frac{(1 - \phi)\bar{\theta}}{1 - \phi} + \frac{\phi\bar{\theta} - \phi\theta}{1 - \phi} \]
  \[ = \bar{\theta} + \frac{\phi}{1 - \phi} \Delta \theta, \]
  where we define $\Delta \theta := \bar{\theta} - \theta$
• Since $\phi \in (0, 1)$ and $s$ is concave (i.e., $s'(q)$ is decreasing in $q$) it holds that $\bar{q}^* < \bar{q}^{FB}$
• Due to $s'(0) = \infty$ it also holds that $\bar{q}^* > 0$
Optimal transfers

- We already know that $\bar{t}^* = \bar{\theta}\bar{q}^*$ and $t^* = \theta q^* + \bar{t}^* - \theta \bar{q}^*$
- The optimal transfers are hence
  \[ \bar{t}^* = \bar{\theta}\bar{q}^* \text{ and } t^* = \theta q^* + \bar{q}^* \Delta \theta \]
- Since $\bar{q}^* < \bar{q}^{FB} < q^{FB} = q^*$, it holds that $\bar{q}^* < q^*$
- This implies that $\bar{t}^* < t^*$
- That is, the principal optimally suggests a menu of contracts which consists of
  - one contract that specifies a relatively low quantity $\bar{q}^*$ and a relatively low transfer $\bar{t}^*$ (this contract is designated to the agent if he has a high-cost type $\bar{\theta}$) and
  - another contract that specifies a relatively high quantity $q^*$ and a relatively high transfer $t^*$ (this contract is designated to the agent if he has a low-cost type $\theta$)
Was it justified to neglect $(IC)$?

- Since $(IC)$ is binding, $\Delta t^* = \theta \Delta q^*$, where we define $\Delta t^* := \bar{t}^* - \underline{t}^*$ and $\Delta q^* := \bar{q}^* - \underline{q}^*$

- The incentive constraint $(IC)$ writes $\Delta t^* \geq \bar{\theta} \Delta q^*$

- Since $\Delta q^* < 0$, it holds that $\Delta t^* = \underline{\theta} \Delta q^* > \bar{\theta} \Delta q^*$

- It was hence indeed justified to neglect $(IC)$
Proposition 3

With asymmetric information, the principal optimally sets the menu of contracts \( \{ (q^*, t^* = \theta q^* + \tilde{q}^* \Delta \theta), (\tilde{q}^*, \tilde{t}^* = \tilde{\theta} \tilde{q}^*) \} \), where \( \tilde{q}^* \) solves \( s'(q) = \tilde{\theta} + \frac{\phi}{1-\phi} \Delta \theta \). It holds that \( q^{FB} = q^* > \tilde{q}^{FB} > \tilde{q}^* > 0 \).

- **Interpretation:**
  - Asymmetric information cause that the principal does not implement the first-best quantities (=efficient quantities) for all types.
  - There are hence welfare losses, i.e., a market failure.

- **Example:**
  - Suppose \( s(q) = \sqrt{q}, \theta = 1, \tilde{\theta} = 2, \) and \( \phi = 1/2 \)
  - The principal optimally sets the menu of contracts
    \( \{ (q^* = \frac{1}{4}, t^* = \frac{5}{18}), (\tilde{q}^* = \frac{1}{36}, \tilde{t}^* = \frac{1}{18}) \} \)
Utilities

- The agent’s utility if he has type $\bar{\theta}$ is
  \[ \bar{u}_A^* = \bar{t}^* - \bar{\theta}\bar{q}^* = 0, \]
  where the last equality follows since ($\text{PC}$) is binding

- If the agent has type $\theta$ it is
  \[ u_A^* = t^* - \theta q^* = \bar{t}^* - \theta \bar{q}^* = \bar{q}^* \Delta \theta > 0, \]
  where the second equality follows since ($\text{IC}$) is binding and the third equality follows from $\bar{t}^* = \bar{\theta}\bar{q}^*$

- Thus, if the agent is of the high-cost type, he experiences a utility which just equals his reservation utility

- In contrast, if the agent is of the low-cost type, his utility exceeds his reservation utility, i.e., he receives an information rent

- Intuition: Since the low-cost type can choose the same contract as the high-cost type, he must be better off than the high-cost type for any positive quantity $\bar{q}$ that is implemented
Who benefits from asymmetric information?

- The agent benefits from asymmetric information, since his expected utility is zero in case of symmetric information (recall that all types receive a transfer that just covers the costs), while his expected utility is positive in case of asymmetric information (recall that the agent receives an information rent if he is of the low-cost type).

- The principal suffers from asymmetric information, since he cannot implement the same contracts in case of asymmetric information as in case of symmetric information, which implies that her expected utility is lower with asymmetric information than with symmetric information.

- Since asymmetric information cause that the principal does not implement the first-best quantities for all types, the expected welfare is lower with asymmetric information than with symmetric information (this result further implies that the principal suffers more from asymmetric information than the agent benefits from them).
Why does the principal not implement the FB?

- The principal faces a trade-off between rent extraction vs. efficiency
  - The quantity $\bar{q} = 0$ extracts all of the low-cost agent’s rent (recall that $u^*_A = \bar{q}^* \Delta \theta$, which is zero for $\bar{q} = 0$)
  - The quantity $\bar{q} = \bar{q}^{FB}$ maximizes efficiency
  - Due to the trade-off, the principal optimally implements a quantity $\bar{q}$ between 0 and $\bar{q}^{FB}$

- Technically, implementing a large quantity $\bar{q}$ is double costly: the principal has to pay the high-cost type his production costs of $\bar{\theta} \bar{q}$ and the low-cost type a large transfer $t = \theta q + \bar{q} \Delta \theta$
More than two types

- Suppose $\theta \in \{\theta_1, \ldots, \theta_n\}$, with $n > 2$ and we order the types such that $\theta_i > \theta_{i-1}$ for all $i > 1$

- Each type has one participation constraint and $n - 1$ incentive constraints

- In total, there are hence $n$ participation constraints and $n(n - 1)$ incentive constraints

- Fortunately, we only have to consider the participation constraint of the type with the highest costs and one incentive constraint for each type (the constraint that he does not want to choose the contract designated to the neighboring higher cost type)
Summary

- With symmetric information (agent and principal know agent’s type), i.e., the first-best: the efficient quantity is implemented for all types and there are no types that have an information rent.

- With asymmetric information (only the agent knows his type): the first-best quantity is only implemented for the most productive type, and all but the least productive type have an information rent.

- Adverse selection models have many applications: supply contracts, banking, insurance, regulation, taxation, auctions, bargaining over measures to protect the climate ... (see Laffont and Martimort 2001 and Salanié 2005)

- With asymmetric information, we should thus not expect that the parties achieve an efficient solution.
Exercises
Exercise 2.1

- Suppose $s(q) = 2\sqrt{q}$, $\bar{\theta} = 2$, and $\theta = 1$
- Determine the first-best and the optimal contracts for the case of symmetric information
- Show that these contracts cannot be implemented with asymmetric information
Solution Exercise 2.1

- With symmetric information the principal’s problem is to maximize her utility \( u_P = 2\sqrt{q} - t \) subject to the agent’s participation constraint \( t - \theta q \geq 0 \), where \( \theta \) is either \( \bar{\theta} = 2 \) or \( \underline{\theta} = 1 \)

- We can write the problem as a Lagrangian:

\[
L(q, t) = 2\sqrt{q} - t + \lambda(t - \theta q)
\]

- In the optimum, the following holds:

\[
\frac{\partial L(\cdot)}{\partial t} = -1 + \lambda = 0 \quad (10)
\]

\[
\frac{\partial L(\cdot)}{\partial q} = q^{-1/2} - \lambda\theta = 0 \quad (11)
\]

\[
t - \theta q \geq 0 \quad (PC)
\]

\[
\lambda \geq 0; \quad \text{if } t - \theta q > 0, \text{ then } \lambda = 0 \quad (12)
\]

- Remark: The complementary slackness condition “if \( t - \theta q > 0 \), then \( \lambda = 0 \)” can also be written as \( \lambda(t - \theta q) = 0 \)
Solving the system

- From (10) we get that
  \[ \lambda^{FB} = 1 \]

- Since \( \lambda^{FB} = 1 \), the (PC) binds and so
  \[ t^{FB} = \theta q^{FB} \]

- Plugging \( \lambda^{FB} = 1 \) into (11) yields
  \[ q^{FB} = \frac{1}{\theta^2} \]

- Hence, \( q^{FB} = 1 \) and \( \bar{q}^{FB} = 1/4 \)

- The optimal contract is \((q^{FB} = 1, t^{FB} = \theta q^{FB} = 1)\) in case \( \theta = \theta \)
  and \((\bar{q}^{FB} = 1/4, \bar{t}^{FB} = \bar{\theta} \bar{q}^{FB} = 1/2)\) in case \( \theta = \bar{\theta} \)
Non-implementability

- With asymmetric information, type $\theta$ receives utility 0 if he chooses the contract $(q_{FB}^{FB} = 1, t_{FB}^{FB} = \theta q_{FB}^{FB} = 1)$, while his utility is $1/2 - 1 \times 1/4 = 1/4$ if he chooses the contract $(\bar{q}_{FB}^{FB} = 1/4, \bar{t}_{FB}^{FB} = \bar{\theta} \bar{q}_{FB}^{FB} = 1/2)$.

- Hence, the contracts that are optimal in case of symmetric information are not incentive compatible and can thus not be implemented with asymmetric information.
Exercise 2.2

- Suppose again that \( s(q) = 2\sqrt{q} \), \( \bar{\theta} = 2 \), and \( \theta = 1 \)
- But now there is asymmetric information (only the agent knows his type)
- Let \( \phi \) be either 0, 1/3, 1/2, or 2/3
- Determine the optimal menu of contracts for the different values of \( \phi \)
Since $s(q) = 2\sqrt{q}$, we have that $s'(q) = q^{-1/2}$

From the formulas of the lecture we get that the optimal quantities are $q^* = 1$ and $\bar{q}^* = \frac{1}{(2 + \frac{\phi}{1-\phi})^2}$

The optimal transfers are $\underline{t}^* = 1 + \bar{q}^*$ and $\overline{t}^* = 2\bar{q}^*$

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<td>$\frac{2}{6.25}$</td>
<td>$\frac{2}{9}$</td>
<td>$\frac{2}{16}$</td>
</tr>
</tbody>
</table>

Hint: in the exam, you have to derive all formulas by using Lagrangians
Exercise 2.3

- Suppose there is asymmetric information
- How is $\tilde{q}^*$ related to $\phi$?
- What is the intuition?
Solution Exercise 2.3

- $\bar{q}^*$ is determined by $s'(\bar{q}^*) = \bar{\theta} + \frac{\phi}{1-\phi} \Delta \theta$

- Since $\frac{\partial \phi}{\partial \phi} = \frac{1-\phi-\phi \times (-1)}{(1-\phi)^2} = \frac{1}{(1-\phi)^2} > 0$, the right-hand side is increasing in $\phi$

- Because $s$ is concave (i.e., $s'(q)$ is decreasing in $q$), $\bar{q}^*$ is decreasing in $\phi$

- In particular, for $\phi \to 0$, $\bar{q}^* \to \bar{q}^{FB}$, while for $\phi \to 1$, $\bar{q}^* \to 0$
Intuition

• If it turns out that the agent is of type $\tilde{\theta}$, then it is good for the principal to have a quantity that is as close as possible to $\bar{q}^{FB}$, i.e., a relatively high quantity $\bar{q}$ (cf. the case where the principal knows the agent’s type)

• If it turns out that the agent is of type $\theta$, then it is good for the principal to have a low quantity $\bar{q}$, since the transfer $t$ is increasing in $\bar{q}$

• If the probability that the agent has type $\theta$, i.e., $\phi$, is higher, the latter aspect gets more important, while the former aspect gets less important

• It is hence optimal to implement a lower quantity $\bar{q}^*$ if $\phi$ is higher
Exercise 2.4

- Suppose \( s(q) = \beta \left( 1 - \frac{1}{1+q} \right) \), \( \phi = 2/3 \), \( \bar{\theta} = 2 \), and \( \theta = 1 \)
- Let \( \beta \) be either 4 or 16
- Suppose there is asymmetric information
- Determine the optimal quantities \( \bar{q}^* \) and \( q^* \) for the different values of \( \beta \) and interpret your results
Solution Exercise 2.4

- $s(q) = \beta(1 - (1 + q)^{-1}) \implies s'(q) = \beta (1 + q)^{-2}$
- $s'(\underline{q}^*) = \theta \implies \beta (1 + \underline{q}^*)^{-2} = 1 \implies (1 + \underline{q}^*)^{-2} = 1/\beta \implies 1 + \underline{q}^* = (1/\beta)^{-1/2} \implies \underline{q}^* = \beta^{1/2} - 1$
- $s'(\bar{q}^*) = \bar{\theta} + \frac{\phi}{1-\phi}\Delta \theta \implies \beta (1 + \bar{q}^*)^{-2} = 2 + 2 \implies (1 + \bar{q}^*)^{-2} = 4/\beta \implies (1 + \bar{q}^*) = (4/\beta)^{-1/2} \implies \bar{q}^* = (4/\beta)^{-1/2} - 1 = (\beta/4)^{1/2} - 1$
- For $\beta = 4$ we have $\underline{q}^* = 1$ and $\bar{q}^* = 0$
- Note that the reason for $\bar{q}^* = 0$ is that in this exercise $s'(0)$ is finite (vs. lecture)
- For $\beta = 16$ we have $\underline{q}^* = 3$ and $\bar{q}^* = 1$
- Interpretation: The higher is $\beta$, the higher is the principal’s benefit; hence, for a higher $\beta$, the optimal quantities $\bar{q}^*$ and $\underline{q}^*$ are higher $\Box$
Exercise 2.5

- Take the values from Exercise 2.2, let $\phi = 1/2$, and suppose that the principal’s reservation utility $u^{res}_P$ is not necessarily zero.
- Can the principal improve by designing a contract which the agent only accepts if he has a low-cost type $\theta$?
Solution Exercise 2.5

- From Exercise 2.2 we know that for $\phi = 1/2$ the optimal menu of contracts is \( \{(q^* = 1, t^* = 1\frac{1}{9}), (\bar{q}^* = 1/9, \bar{t}^* = 2/9)\} \)

- The principal’s expected utility is hence

\[
E[u^*_P] = \phi \left(s(q^*) - t^*\right) + (1 - \phi) \left(s(\bar{q}^*) - \bar{t}^*\right)
= 1/2 \left(2\sqrt{q} - t\right) + 1/2 \left(2\sqrt{\bar{q}} - \bar{t}\right)
= 1/2 \left(2\sqrt{1} - 10/9\right) + 1/2 \left(2\sqrt{1/9} - 2/9\right)
= 2/3
\]
One contract

- Suppose the principal alternatively designs just a contract for the type $\theta$ (which the type $\bar{\theta}$ does not accept)
- The principal’s problem is then to choose $q$ and $t$ to maximize her expected utility

$$\phi(s(q^*) - t^*) + (1 - \phi)u_P^{res}$$

subject to the participation constraint of type $\theta$

$$t^* - \theta q^* \geq 0 \quad (PC)$$

and the constraint that type $\bar{\theta}$ does not accept the contract

$$t^* - \bar{\theta} q^* < 0$$

- We neglect the latter constraint because we are confident that it will not bind and later verify that this is true
• It is straightforward that (as in case the principal knows that the agent is of type $\theta$) that $q^* = q^{FB} = 1$ and the transfer is such that the participation constraint holds with equality, $t^* = 1$

• Hence, type $\bar{\theta}$ indeed does not want to participate ($t^* - \bar{\theta} q^* = 1 - 2 < 0$)

• The principal’s expected utility is hence

$$E[u^*_P] = \phi(2\sqrt{1} - 1) + (1 - \phi)u^*_P^{res} = 1/2 + u^*_P^{res}/2$$

• Thus, for

$$1/2 + u^*_P^{res}/2 > 2/3 \iff u^*_P^{res} > 1/3,$$

the principal prefers to design just a contract for the type $\theta$

• Note that the same holds true if $u^*_P^{res}$ is kept fixed at zero, but $u^*_A^{res}$ is sufficiently high
Exercise 2.6

- Take the values from Exercise 2.2 and let $\phi = 1/2$
- Compare the per-unit cost of the two types, i.e., $t^*/q^*$ and $\bar{t}^*/\bar{q}^*$
- Does this comparison hold in general?
Solution Exercise 2.6

- From Exercise 2.2 we know that $q^* = 1$, $t^* = 1\frac{1}{9} = 10/9$, $\bar{q}^* = 1/9$, and $\bar{t}^* = 2/9$
- The per-unit costs are $\frac{t^*}{q^*} = \frac{10/9}{1} = \frac{10}{9}$ and $\frac{\bar{t}^*}{\bar{q}^*} = \frac{2/9}{1/9} = 2$
- The per-unit costs are hence lower if the agent has type $\theta$ than if he has type $\bar{\theta}$
General insight?

- Since \((\overline{PC})\) is binding, we can rewrite \(\bar{t}^* - \bar{\theta}\bar{q}^* = 0\) as
  \[
  \frac{\bar{t}^*}{\bar{q}^*} = \bar{\theta}
  \]

- Since \((\overline{IC})\) is binding, we can rewrite \(t^* - \theta q^* = \bar{t}^* - \bar{\theta}\bar{q}^*\) as
  \[
  \frac{t^*}{q^*} = \frac{\theta q^* + \bar{t}^* - \theta \bar{q}^*}{q^*}
  \]

- Using that \((\overline{PC})\) is binding, we can rearrange to
  \[
  \frac{t^*}{q^*} = \frac{\theta q^* + \bar{\theta}q^* - \theta \bar{q}^*}{q^*} = \frac{\theta q^* + \Delta \theta \bar{q}^*}{q^*} = \theta + \Delta \theta \frac{\bar{q}^*}{q^*} < \theta + \Delta \theta = \bar{\theta}
  \]

- Hence, \(\frac{t^*}{q^*} < \frac{\bar{t}^*}{\bar{q}^*}\) holds also in the general case \(\square\)
Exercise 2.7

A municipality has forbidden traffic to enter its city center. The municipality suggests that its inhabitants take the metro. The management of the metro is given to a monopoly firm which is responsible for the pricing of the metro tickets. A market analysis reveals that there are two types of travelers. The willingness to pay for travelers of type 1 is $4\sqrt{q}$ for $q$ tickets. For travelers of type 2, the willingness to pay is $3\sqrt{q}$. There are .5 million travelers of each type. Providing one additional ticket has a constant cost of .5 Euros. The monopoly sell multi-journey tickets (books of tickets).

- Determine the efficient quantities
- Determine the optimal menu of contracts for the case where the monopoly cannot distinguish between type 1 and a type 2 travelers
- Hints: the principal delivers something to the agent and the utility functions are different than before
Solution Exercise 2.7

- \( u_A = \theta \sqrt{q} - t \), where \( \theta \in \{\theta = 3, \bar{\theta} = 4\} \) and \( t \) is now the transfer to the principal
- Principal’s utility per traveler is \( u_P = t - .5q \)
- We maximize the welfare \( \theta \sqrt{q} - .5q \)
- Differentiating with respect to \( q \) yields
  \[
  \theta \cdot .5q^{-1/2} - .5 = 0
  \]
- Hence, \( q^{effi} = \theta^2 \) and so \( q^{effi} = 9, \bar{q}^{effi} = 16 \)
Asymmetric information

- The incentive constraints are
  \[3\sqrt{q - t} \geq 3\sqrt{\bar{q} - \bar{t}}\] \hspace{1cm} (IC)
  \[4\sqrt{q - \bar{t}} \geq 4\sqrt{q - t}\] \hspace{1cm} (IC)

- The participation constraints are
  \[3\sqrt{q - t} \geq 0\] \hspace{1cm} (PC)
  \[4\sqrt{\bar{q} - \bar{t}} \geq 0\] \hspace{1cm} (PC)
Neglect some constraints

- Since the high-utility type $\theta = \bar{\theta} = 4$ can choose the contract designated to the low-utility type $\theta = \theta = 3$, his participation constraint does not bind (it is implied by ($IC$) and ($PC$))
- We hence neglect ($PC$)
- We suppose that the low-utility type does not want to mimic the high-utility type
- We hence neglect the incentive constraint ($IC$)
- Note that this is vice-versa than in the models considered before
- The Lagrangian writes

$$L(q, t, \bar{q}, \bar{t}) = .5 \left( t - .5q \right) + .5 \left( \bar{t} - .5\bar{q} \right)$$

$$+ \lambda \left( 3\sqrt{q} - t \right) + \bar{\mu} \left( 4\sqrt{\bar{q}} - \bar{t} - 4\sqrt{\bar{q} + t} \right)$$
First-order conditions

\[ \frac{\partial L(\cdot)}{\partial \bar{t}} = .5 - \bar{\mu} = 0 \]

Hence, \( \bar{\mu}^* = .5 \) and thus \((\text{IC})\) is binding, i.e., holds with \( = \)

\[ \frac{\partial L(\cdot)}{\partial t} = .5 - \lambda + \bar{\mu} = 0 \]

Hence, \( \lambda^* = 1 \) and thus \((\text{PC})\) is binding

\[ \frac{\partial L(\cdot)}{\partial q} = - .25 + \lambda 1.5q^{-1/2} - \bar{\mu}2q^{-1/2} = 0 \]

Thus, \( q^* = 4 \)

\[ \frac{\partial L(\cdot)}{\partial \bar{q}} = - .25 + \bar{\mu}2\bar{q}^{-1/2} = 0 \]

Thus, \( \bar{q}^* = 16 \)
Optimal contract

- Since $(PC)$ and $(IC)$ are binding, we obtain the optimal transfers: $t^* = 6$ and $\bar{t}^* = 14$

- It is also readily checked that the neglected constraints, $(PC)$ and $(IC)$, are indeed satisfied

- The optimal menu of contracts is hence

  $$ \{(q^* = 4, t^* = 6), (\bar{q}^* = 16, \bar{t}^* = 14)\} $$
Exercise 2.8

- A wine seller can produce vines in different qualities
- Her production costs per bottle of quality $q \geq 0$ are $q^2$
- There are two types of consumers: modest consumers of type $\theta$ and sophisticated consumers with type $\bar{\theta}$, where $0 < \theta < \bar{\theta}$
- A consumer’ utility from a bottle wine of quality $q$ is $\theta q$ and 0 if he does not buy wine
- Each consumer buys at most one bottle of wine
- The wine seller knows that there are $n > 0$ modest consumers and $\bar{n} > 0$ sophisticated consumers
- Determine the efficient quantities and the optimal contracts with symmetric information (where she knows each consumers’ type) and with asymmetric information
Solution Exercise 2.8

- \( u_A = \theta q - t \), where \( \theta \in \{ \underline{\theta}, \bar{\theta} \} \) and \( t \) is the transfer to the principal
- Wine seller’s utility per consumer is \( u_P = t - q^2 \)
- We maximize the welfare \( \theta q - q^2 \)
- Differentiating with respect to \( q \) yields
  \[ \theta - 2q = 0 \]
- Hence, \( q^{\text{effi}} = \theta / 2 \) and so \( \underline{q}^{\text{effi}} = \underline{\theta} / 2 \), \( \bar{q}^{\text{effi}} = \bar{\theta} / 2 \)
Symmetric Information

- The wine seller/principal knows each consumer’s type
- Facing consumer with type $\theta$, the Lagrangian writes
  \[ L(q, t) = t - q^2 + \lambda(\theta q - t) \]

- The first-order conditions are
  \[ \frac{\partial L(q, t)}{\partial t} = 1 - \lambda = 0 \]
  Hence, $\lambda^{FB} = 1$ and so the participation constraint is binding, implying $t^{FB} = \theta q^{FB}$

- Therefore, she optimally offers each consumer with type $\theta$ the contract $(q^{FB} = \theta/2, t^{FB} = \theta^2/2)$ and each consumer with type $\bar{\theta}$ the contract $(\bar{q}^{FB} = \bar{\theta}/2, \bar{t}^{FB} = \bar{\theta}^2/2)$
Asymmetric information

- The incentive constraints are

\[
\theta q - t \geq \theta \bar{q} - \bar{t} \quad (IC)
\]

\[
\theta \bar{q} - \bar{t} \geq \theta q - t \quad (\overline{IC})
\]

- The participation constraints are

\[
\theta q - t \geq 0 \quad (PC)
\]

\[
\theta \bar{q} - \bar{t} \geq 0 \quad (\overline{PC})
\]
Neglect some constraints

- Since the high-utility type $\theta = \bar{\theta}$ can choose the contract designated to the low-utility type $\theta = \underline{\theta}$, his participation constraint does not bind (it is implied by (IC) and (PC))

- We hence neglect (PC)

- We suppose that the low-utility type does not want to mimic the high-utility type

- We hence neglect the incentive constraint (IC)

- The Lagrangian writes

$$
\mathcal{L}(q, t, \bar{q}, \bar{t}) = n(t - q^2) + \bar{n}(\bar{t} - \bar{q}^2) \\
+ \lambda(\theta q - t) + \bar{\mu}(\bar{\theta} \bar{q} - \bar{t} - \bar{\theta} q + t)
$$
First-order conditions

\[ \frac{\partial L(\cdot)}{\partial \bar{t}} = \bar{n} - \bar{\mu} = 0 \]

Hence, \( \bar{\mu}^* = \bar{n} > 0 \) and thus \((IC)\) is binding

\[ \frac{\partial L(\cdot)}{\partial \bar{t}} = n - \lambda + \bar{\mu} = 0 \]

Hence, \( \lambda^* = n + \bar{n} > 0 \) and thus \((PC)\) is binding

\[ \frac{\partial L(\cdot)}{\partial q} = -n2q + \lambda\theta - \bar{\mu}\bar{\theta} = 0 \]

Thus, \( q^* = \frac{\theta}{2} - \frac{A\theta}{2} \frac{\bar{n}}{\bar{n}} < q^{FB} \)

\[ \frac{\partial L(\cdot)}{\partial \bar{q}} = -\bar{n}2\bar{q} + \bar{\mu}\bar{\theta} = 0 \]

Thus, \( \bar{q}^* = \bar{\theta}/2 = \bar{q}^{FB} \)
Optimal contract

Since $(PC)$ and $(IC)$ hold with $=$, we obtain the optimal transfers:

\[
t^\ast = \theta q^\ast = \frac{\theta^2}{2} - \frac{\theta \Delta \theta}{2} \bar{n} \]

\[
\bar{t}^\ast = \bar{\theta} \bar{q}^\ast - \Delta q^\ast = \frac{\bar{\theta}^2}{2} - \frac{\theta \Delta \theta}{2} + \frac{(\Delta \theta)^2}{2} \bar{n}
\]

It is also readily checked that the neglected constraints, $(PC)$ and $(IC)$, are indeed satisfied

The optimal menu of contracts is hence

\[
\left\{ \left( q^\ast = \frac{\theta}{2} - \frac{\Delta \theta}{2} \bar{n}, t^\ast = \frac{\theta^2}{2} - \frac{\theta \Delta \theta}{2} \bar{n} \right), \left( \bar{q}^\ast = \frac{\bar{\theta}}{2}, \bar{t}^\ast = \frac{\bar{\theta}^2}{2} - \frac{\theta \Delta \theta}{2} + \frac{(\Delta \theta)^2}{2} \bar{n} \right) \right\}
\]
Remark on total number of consumers

- Denote the total number of consumers by \( n \), i.e., \( n = \underline{n} + \bar{n} \)
- Denote the share of consumers of type \( \theta \) by \( \phi := \frac{n}{\underline{n}} \)
- The share of consumers of type \( \bar{\theta} \) is then \( 1 - \phi = \bar{n}/n \)
- Since

\[
\frac{\bar{n}}{\underline{n}} = \frac{\bar{n}/n}{n/n} = \frac{1 - \phi}{\phi}
\]

the optimal menu of contracts does not depend on the total number of consumers \( n \)

\( \square \)
Exercise 2.9

- A lender can provide a loan of size \( k \) to a borrower.

- Lender’s utility is \( u_L = t - Rk \), where \( R \) is the risk-free interest rate.

- Borrower’s utility is \( \theta f(k) - t \), where \( \theta f(k) \) is the production with \( k \) units of capital and \( f' > 0 \) and \( f'' < 0 \).

- Productivity \( \theta \) is \( \bar{\theta} \) with probability \( \phi \) and \( \underline{\theta} \) with probability \( 1 - \phi \).

- Determine the efficient loans \( k^{\text{effi}} \) and \( \bar{k}^{\text{effi}} \).

- What are the optimal loans \( k^* \) and \( \bar{k}^* \) if the lender does not know the borrower’s type?
Solution Exercise 2.9

- The efficient loan $k^{effi}$ solves $\theta f'(k) = R$
- Interpretation: the return on capital is equal to the risk free interest rate
- It is readily shown that with asymmetric information there is efficient capital provision to high-productivity type: $\bar{k}^* = \bar{k}^{effi}$
- However, there is underprovision of capital to the low-productivity type: $k^* < \bar{k}^{effi}$, since $k^*$ solves

$$\left(\theta - \frac{\phi}{1 - \phi} \Delta \theta\right) f'(k) = R$$
Exercise 2.10

- Suppose \( u_A = t - \theta q \), \( u_P = 2\sqrt{q} - t \), where \( \theta \in \{ \theta_1 = 1, \theta_2 = 2, \theta_3 = 3 \} \) and \( \phi_1 = \phi_2 = \phi_3 = \frac{1}{3} \)
- Determine the efficient quantities \( q_1^{\text{effi}} \), \( q_2^{\text{effi}} \), and \( q_3^{\text{effi}} \)
- Determine the optimal menu of contracts when there is asymmetric information
Solution Exercise 2.10

- I will only provide the results
- \( q_1^{\text{effi}} = 1, \quad q_2^{\text{effi}} = \frac{1}{4}, \quad \text{and} \quad q_3^{\text{effi}} = \frac{1}{9} \)
- \( q_1^* = 1, \quad t_1^* = \frac{259}{225}, \quad q_2^* = \frac{1}{9}, \quad t_2^* = \frac{59}{225}, \quad q_3^* = \frac{1}{25}, \quad t_3^* = \frac{3}{25} \)
Exercise 2.11

- Imagine you run a train company in the nineteenth century who offers three classes.
- There are three types of train passengers: rich (who should use the first class), middle class (who should use the second class), and poor (who should use the third class).
- Third-class wagons could be ordered with or without a roof.
- The costs for both types of wagons is the same.
- Which type of wagons should the company use?
- Discuss your results with respect to the passengers’ incentive and participation constraints.
Solution Exercise 2.11

- Having wagons without roofs for the third class makes it more difficult that poor passengers buy a train ticket; their participation constraint gets more demanding.

- However, wagons without roofs for the third class makes the second class more attractive for the middle class relative to the third class; the incentive constraint of the middle class gets less demanding.

- It could thus be optimal to have wagons without roofs, even if they cost the same as wagons with roofs.