3 Moral Hazard

Information asymmetries concerning actions
Motivation

- Agents often have some freedom to choose actions that the principal cannot or not perfectly observe.

Examples:
- An agent that works for a principal could decide how hard to work.
- An agent that is insured by a principal may take little care.

If this is true, the agent may exploit his freedom to choose actions opportunistically, that is, actions that benefit his utility, but may harm the principal.
3.1 Basic Model
Basic idea

- Suppose that some outcome, e.g., the principal’s return, the output, or some other measure of performance, is contractible
  - This is plausible for many, though not all, performance measures: e.g., a manager’s remuneration can depend on the firm’s profit or a doctor’s remuneration on the healing rate of his patients
- Then the principal is able to write a contract that specifies the agent’s remuneration in dependence of his performance
- This could motivate the agent to take actions that are in the principal’s interest
Basic model

- There is a finite number $M$ of possible outcomes.
- The outcome $m \in \{1, \ldots, M\}$ is associated with a return $r_m$ for the principal, where we order the outcomes such that $r_{m+1} > r_m$ for all $m \in \{1, \ldots, M - 1\}$.
- The return depends on effort $e$ and the realization of a noise term $\varepsilon$: $r = f(e, \varepsilon)$.
- We use this notation in the LEN model below.
- In the other models, the following notation is more convenient: The probability that outcome $m$ (with a return $r_m$) realizes is $p_m(e)$.
- It holds that $\sum_{m=1}^{M} p_m(e) = 1$ for all efforts $e$.
- We suppose that $p_m(e) \in (0,1)$ and that $p'_m(e) \neq 0$ for all efforts and outcomes.
Effort

Effort is a continuous, nonnegative decision variable of the agent: 
\[ e \in \mathbb{R}_{\geq 0} \]

The effort costs \( c(e) \) satisfy the following assumptions: \( c \) is twice continuously differentiable, increasing, \( c'(e) > 0 \) for all \( e > 0 \), and convex, \( c''(e) > 0 \), and satisfies \( c(0) = c'(0) = 0 \) and 
\[ \lim_{e \to \infty} c'(e) = \infty \]

- Simple example: \( c(e) = e^2 \)

We assume that effort is productive in the sense that the expected return \( E[r] = \sum_{m=1}^{M} p_m(e)r_m \) is increasing in the effort level: 
\[ \frac{\partial E[r]}{\partial e} = \sum_{m=1}^{M} p'_m(e)r_m > 0 \]
Remuneration

- The agent’s remuneration can depend on the outcome, but not on effort (moral hazard)
- The agent receives the wage \( w = w_m \) in case outcome \( m \) realizes
- A contract thus takes the form \( (w_1, \ldots, w_M) \)
Timing

1. The principal suggests a contract
2. The agent can either accept or reject
   - If the agent rejects, he receives the reservation utility $u_{res}^A$
   - If the agent accepts, he chooses an effort level
3. The outcome is determined and the payment is made according to the contract
Principal’s utility

- The principal is risk neutral
- Her utility is
  \[ u_P = r - w \]
- Assuming risk neutrality on the side of the principal is common
- The usual justification is that principals often have many agents and risks. Due to diversification, they do not care about the risk they have in the relationship to an agent
Simplification and assumptions

- Consider the case with two outcomes, \( M = 2 \)
  - We can then interpret outcome 2 as high outcome, high performance, or a success
  - And outcome 1 as low outcome, low performance, or a failure
  - We can write \( p(e) \) for \( p_2(e) \) and, since \( \sum_{m=1}^{M} p_m(e) = 1 \), \( 1 - p(e) \) for \( p_1(e) \)
- The assumption \( \sum_{m=1}^{M} p'_m(e) r_m > 0 \) implies that \( p'(e) > 0 \)
- That is, effort increases the probability that the outcome is high, i.e., 2, such that the high return \( r_2 \) is realized
Moreover, we suppose that $p''(e) \leq 0$

- This can be interpreted that the marginal productivity is at least weakly decreasing
- That is, for low [high] levels of effort, an additional marginal unit of effort increases the success probability relatively much [little]

Simple examples: $p(e) = \alpha e + \beta$, with $\alpha > 0$ and $\beta \geq 0$, or $p(e) = \sqrt{e}$
3.2 Risk Neutrality
Agent’s utility

- Suppose that also the agent is risk neutral

- His utility function is

\[ u_A = w - c(e) \]

- The agent’s expected utility is

\[
E[u_A] = \sum_{m=1}^{M} p_m(e)u_A \\
= p(e)(w_2 - c(e)) + (1 - p(e))(w_1 - c(e)) \\
= p(e)w_2 + (1 - p(e))w_1 - c(e)
\]
Agent’s effort choice

- Given some contract \((w_1, w_2)\), the agent maximizes his expected utility over his effort choice.
- If the principal desires to implement effort \(\hat{e}\), this must also be the effort level the agent likes to choose:
  \[
  \hat{e} \in \arg\max E[u_A]
  \]  
  (IC)

- This condition is called incentive constraint.

- Remark: In principle, several efforts could maximize \(E[u_A]\):
  - This is why we write \(\hat{e} \in \arg\max E[u_A]\) and not \(\hat{e} = \arg\max E[u_A]\).
  - In the models we consider only one effort level maximizes \(E[u_A]\).
The first-order condition of the agent’s problem is

\[
\frac{\partial E[u_A]}{\partial e} = p'(e)w_2 - p'(e)w_1 - c'(e) = 0
\]

\[\iff p'(e)\Delta w = c'(e),\]

where \(\Delta w := w_2 - w_1\) is the wage spread.

We can interpret \(\Delta w\) also as the incentive power of the contract.

Note that for \(\Delta w < 0\), \(\frac{\partial E[u_A]}{\partial e} < 0\) for all efforts \(e \geq 0\).

Thus, the agent then optimally chooses \(e = 0\).
Suppose next that $\Delta w \geq 0$

The second-order condition is
\[
\frac{\partial^2 E[u_A]}{\partial e^2} = p''(e)\Delta w - c''(e)
\]

Due to the assumptions we made on the functions $p$ and $c$
\[
\frac{\partial^2 E[u_A]}{\partial e^2} < 0
\]

Therefore, the agent’s optimal effort choice is given by the first-order condition
\[
\frac{\partial E[u_A]}{\partial e} = p'(e)\Delta w - c'(e) = 0 \quad (IC')
\]

For example, if $c(e) = \frac{e^2}{2}$ and $p(e) = e$, with $e \in [0,1]$, then (IC') can be written as $\Delta w = e$
Effort and incentives

- How is the agent’s effort choice related to the incentive power?
- We concentrate on the case $\Delta w \geq 0$
- Implicitly differentiating (IC’) yields that
  \[
  \frac{\partial e}{\partial \Delta w} = - \frac{p'(e)}{p''(e)\Delta w - c''(e)}
  \]
  which is positive
- Thus, the higher are the incentives $\Delta w$, the higher is the effort level the agent optimally chooses
- Put differently, if the principal seeks to implement a higher effort level, she must set higher incentives $\Delta w$
Incentives and relative wages

- The agent’s effort choice depends only on the relative wages, i.e., the wage spread $\Delta w$.
- This can directly be seen from (IC'): $p'(e) \Delta w = c'(e)$.
- The incentive constraint (IC') is thus a condition about relative wages.
Agent’s participation

- The agent optimally participates – i.e., accepts the contract suggested by the principal – if his expected utility from doing so is at least as high as his reservation utility $u_A^{\text{res}}$.

- Formally, the participation constraint has to hold:
  \[ E[u_A] \geq u_A^{\text{res}}, \quad (PC) \]
  where $E[u_A]$ is evaluated at the agent’s optimal effort choice.

- We can rewrite this as
  \[ p(e)w_2 + (1 - p(e))w_1 - c(e) \geq u_A^{\text{res}} \]
Participation and absolute wages

- In contrast to the incentives constraints (IC') or (IC), the participation constraint depends on absolute wages.
  - We can rewrite $E[u_A] = w_1 + p(e)\Delta w - c(e)$, but then the agent’s expected utility still depends on the base wage $w_1$.

- The participation constraint (PC) is thus a condition about absolute wages.
Principal’s problem

- The principal maximizes her expected utility

\[
E[u_P] = \sum_{m=1}^{M} p_m(e) u_P
\]

\[
= p(e) (r_2 - w_2) + (1 - p(e)) (r_1 - w_1)
\]

subject to

- the agent’s participation constraint

\[
p(e) w_2 + (1 - p(e)) w_1 - c(e) \geq u^*_A, \quad (PC)
\]

- and the incentive constraint that the effort the principal desires to implement \( \hat{e} \) is also the effort level the agent likes to choose

\[
\hat{e} \in \arg\max E[u_A] \quad (IC)
\]
First-order approach

- The procedure to replace the global incentive constraint (IC) by the first-order condition of the agent’s effort-selection problem, i.e., the local incentive constraint (IC’), is called first-order approach.
- The first-order approach is valid if the solution of (IC’) is the same as the solution of (IC).
  - (IC’) has only a solution if $\Delta w \geq 0$.
  - Given $\Delta w \geq 0$, we know from above that the second-order condition of the agent’s maximization problem is satisfied, which implies that (IC’) and (IC) yield the same solution.
Relaxed problem

- We are confident that $\Delta w \geq 0$, i.e., that the principal rewards success with an at least weakly higher wage than failure
  
  - Reason: If $\Delta w < 0 \iff w_1 > w_2$, the agent chooses zero effort just as with a fixed wage $\Delta w = 0 \Rightarrow$ the principal can never be better off with $\Delta w < 0$ than with $\Delta w = 0$

- We hence neglect the constraint $\Delta w \geq 0$ (we will later see that it indeed holds) and let the principal solve the relaxed problem:

  $$\max_{(w_1, w_2)} E[u_P] \text{ subject to } (PC), (IC')$$
Idea of solving a relaxed problem

- It is sometimes convenient to solve a relaxed problem instead of the full problem.
- The idea is as follows:
  1. We neglect some constraint(s),
  2. solve the relaxed problem, and
  3. then try to show that the neglected constraint(s) is (are) indeed satisfied.

   - If the neglected constraint(s) is (are) satisfied, then the solution of the relaxed problem is also the solution of the full problem and we are done.
   - If the neglected constraint(s) is (are) not satisfied, then the solution of the relaxed problem is not the solution of the full problem and we have to solve the full problem (or try to solve a different relaxed problem).
Lagrangian

\[ \mathcal{L}(e, w_1, w_2) = p(e) (r_2 - w_2) + (1 - p(e)) (r_1 - w_1) \]
\[ + \lambda (p(e)w_2 + (1 - p(e))w_1 - c(e) - u_A^{\text{res}}) \]
\[ + \mu (p'(e)(w_2 - w_1) - c'(e)) \]
In the optimum, the following holds:

\[
\frac{\partial L(e, w_1, w_2)}{\partial w_1} = -1 + p(e) + \lambda(1 - p(e)) - \mu p'(e) = 0 \quad (1)
\]

\[
\frac{\partial L(e, w_1, w_2)}{\partial w_2} = -p(e) + \lambda p(e) + \mu p'(e) = 0 \quad (2)
\]

\[
\frac{\partial L(e, w_1, w_2)}{\partial e} = p'(e)(r_2 - w_2) - p'(e)(r_1 - w_1) \quad (3)
\]

\[
+ \lambda (p'(e)(w_2 - w_1) - c'(e)) + \mu (p''(e)(w_2 - w_1) - c''(e)) = 0
\]

\[p(e)w_2 + (1 - p(e))w_1 - c(e) \geq u^\text{res}_A \quad \text{(PC)}\]

\[p'(e)(w_2 - w_1) = c'(e) \quad \text{(IC')}\]

\[\lambda \geq 0; \text{ if } p(e)w_2 + (1 - p(e))w_1 - c(e) > u^\text{res}_A, \text{ then } \lambda = 0 \quad (4)\]
Solution

- Adding the first-order conditions (1) and (2) yields \( \lambda^* = 1 \) and thus \( \mu^* = 0 \)
- Using this we obtain that
  \[
  \frac{\partial \mathcal{L}(e, w_1, w_2)}{\partial e} = p'(e)(r_2 - r_1) - c'(e) = 0
  \]
- Thus, the optimal effort \( e^* \) solves
  \[
  p'(e)\Delta r = c'(e)
  \]
- The incentive constraint \((IC')\) is
  \[
  p'(e)\Delta w = c'(e)
  \]
- The former two equations imply that the principal optimally sets the wage spread \( \Delta w^* = \Delta r \)
  - Since \( \Delta w^* = \Delta r > 0 \) it was justified to (i) solve the relaxed problem and (ii) use the first-order approach
Because $\lambda^* > 0$, the (PC) must be binding (i.e., must hold with equality) and we can solve, using $\Delta w^* = \Delta r$, for the optimal contract:

$$w_1^* = -p(e^*)\Delta r + c(e^*) + u_A^{res},$$

$$w_2^* = (1 - p(e^*))\Delta r + c(e^*) + u_A^{res}.$$

- The principal’s expected wage costs are thus
  $$E[w] = c(e^*) + u_A^{res}$$

- The principal’s expected utility is hence
  $$E[u_P] = p(e^*)r_2 + (1 - p(e^*))r_1 - c(e^*) - u_A^{res}$$

- The agent’s expected utility is then
  $$E[u_A] = u_A^{res}$$
Why does the (PC) bind in the optimum?

- Suppose first that for the optimal contract \((w_1^*, w_2^*)\) the agent’s expected utility would fall short of the agent’s reservation utility: \(E[u_A] < u_A^{res}\)
  \[\Rightarrow\] This is not optimal, since the agent does not participate

- Suppose next that for the optimal contract \((w_1^*, w_2^*)\) the agent’s expected utility would exceed the agent’s reservation utility: \(E[u_A] > u_A^{res}\)
  \[\Rightarrow\] The principal can then lower all wages by a small positive amount; the agent still participates and chooses the same effort, while the principal’s expected wage costs lowers such that her expected utility increases

- Hence, for the optimal contract \((w_1^*, w_2^*)\) we must have \(E[u_A] = u_A^{res}\). The agent then participates and the principal cannot lower the wages (without causing the agent to reject the contract)
Benchmark I: welfare maximum

- The expected welfare is
  
  \[
  E[V] = E[u_P] + E[u_A] \\
  = p(e)(r_2 - w_2) + (1 - p(e))(r_1 - w_1) \\
  + p(e)w_2 + (1 - p(e))w_1 - c(e) \\
  = p(e)r_2 + (1 - p(e))r_1 - c(e)
  \]

- The social planner maximizes the expected welfare over effort e:
  
  \[
  \frac{\partial E[V]}{\partial e} = p'(e)(r_2 - r_1) - c'(e) = 0
  \]

- The efficient effort level \( e^{effi} \) thus solves
  
  \[
  p'(e)\Delta r = c'(e)
  \]

- Intuition: The efficient effort level is such that the marginal return of effort equals the marginal costs of effort
Benchmark II: contractible effort

- Suppose effort is contractible
- The contract then takes the form \((e, w)\)
- The principal maximizes her expected utility
  \[ p(e)r_2 + (1 - p(e))r_1 - w \]
  subject to the agent’s participation constraint
  \[ w - c(e) \geq u^{res}_A \] (PC)
- The Lagrangian is thus
  \[ \mathcal{L}(e, w) = p(e)r_2 + (1 - p(e))r_1 - w + \lambda (w - c(e) - u^{res}_A) \]
In the optimum, the following holds:

\[
\frac{\partial \mathcal{L}(e, w)}{\partial w} = -1 + \lambda = 0 \quad (5)
\]

\[
\frac{\partial \mathcal{L}(e, w)}{\partial e} = p'(e)(r_2 - r_1) - \lambda c'(e) = 0 \quad (6)
\]

\[
w - c(e) \geq u_A^{\text{res}} \quad \text{(PC)}
\]

\[
\lambda \geq 0; \text{ if } w - c(e) > u_A^{\text{res}}, \text{ then } \lambda = 0 \quad (7)
\]
Solution

- From the first-order conditions (5) we get that $\lambda^{FB} = 1$
- Thus, by (6),
  \[ \frac{\partial \mathcal{L}(e, w)}{\partial e} = p'(e)(r_2 - r_1) - c'(e) = 0 \]
- The first-best effort $e^{FB}$ hence solves
  \[ p'(e)\Delta r = c'(e) \]
- Because $\lambda^{FB} > 0$, the participation constraint must be binding
- The first-best wage is thus
  \[ w^{FB} = c(e^{FB}) + u^{res}_A \]
- The principal’s expected utility is hence
  \[ E[u_P] = E[r] - w^{FB} = p(e^{FB})r_2 + (1 - p(e^{FB}))r_1 - c(e^{FB}) - u^{res}_A \]
Intuition

- The first-best effort level is such that the marginal return of effort equals the marginal costs of effort
  - The efficient effort level is implemented: $e^{FB} = e^{effi}$
- The first-best wage is chosen such that the agent’s participation constraint holds with equality
Proposition 1

Suppose the agent is risk neutral and the agent’s liability is unlimited. Whether effort is not contractible or contractible, the principal always implements the efficient effort level: \( e^* = e^{FB} = e^{effi} \). For this effort level, the marginal return of effort equals the marginal costs of effort. If effort is not contractible, the principal optimally sets the contract:

\[
\begin{align*}
    w_1^* &= -p(e^*)\Delta r + c(e^*) + u_{A}^{\text{res}}, \\
    w_2^* &= (1 - p(e^*))\Delta r + c(e^*) + u_{A}^{\text{res}}.
\end{align*}
\]

If effort is contractible, she sets the contract \( (e^{FB}, w^{FB} = c(e^{FB}) + u_{A}^{\text{res}}) \). The expected wage costs are the same in case effort is contractible and in case effort is not contractible: \( E[w] = c(e^*) + u_{A}^{\text{res}} \). Also the principal’s expected utility is \( E[u_{P}] = r_1 + p(e^*)\Delta r - c(e^*) - u_{A}^{\text{res}} \) in both cases. The participation constraint binds in both cases such that the agent’s expected utility is \( E[u_{A}] = u_{A}^{\text{res}} \) in both cases.
Interpretation

- It is little surprising that the principal implements the efficient effort level in case effort is contractible, since we know that without information problems and no transaction costs, parties bargaining to an efficient solution (Coase Theorem).
- What is more surprising is that the principal implements the efficient effort level also when effort is not contractible.
- Why does the principal implement the efficient effort level also when there are information problems such that effort is not contractible?
Explanation

- If effort is not contractible, the principal has to set incentives to align the agent’s interests with her own interests.
- Incentives are set by spreading wages, i.e., choosing $\Delta w > 0$.
- The wage the agent receives is then risky: he either receives $w_1$ or $w_2$, with $w_1 < w_2$.
- A risk-neutral agent does not suffer from being exposed to risks.
- Thus, the principal does not have to compensate the agent for bearing risks.
- Setting incentives – and thereby aligning interests – is thus costless for the principal.
• It is therefore optimal for the principal to perfectly align the interests by making the agent the residual claimant\(^1\): \(\Delta w^* = \Delta r\)

• Being the residual claimant, the agent chooses the efficient effort level

• Remark 1: A possible interpretation of the optimal contract is that the principal is “selling the firm” to the agent: The agent has to pay a fixed amount to the principal and then receives the return

• Remark 2: A technical interpretation is that since \(\lambda^* = 1\), the principal maximizes the sum of players’ utilities and therefore implements the efficient effort

\(^1\)The agent is the residual claimant, since once he pays the principal a fixed amount, he gets all remaining returns
3.2.1 Risk Neutrality and more than two Outcomes
Suppose there are more than two outcomes, $M > 2$

The expected welfare is

$$E[V] = E[u_P] + E[u_A]$$

$$= \sum_{m=1}^{M} p_m(e)(r_m - w_m) + \sum_{m=1}^{M} p_m(e)(w_m - c(e))$$

$$= \sum_{m=1}^{M} p_m(e)r_m - c(e)$$

Benchmark I: the efficient effort $e^{effi}$ maximizes the expected welfare, i.e., solves

$$\sum_{m=1}^{M} p'_m(e)r_m = c'(e)$$

Benchmark II: if effort is contractible, the effort $e^{FB} = e^{effi}$ is implemented and the wage is $w^{FB} = c(e^{FB}) + u^{res}_A$
Effort not contractible

- Suppose now that effort is not contractible
- The agent’s expected utility is
  \[ E[u_A] = \sum_{m=1}^{M} p_m(e)(w_m - c(e)) = \sum_{m=1}^{M} p_m(e)w_m - c(e) \]
- The incentive constraint (IC) is \( \hat{e} \in \text{argmax} \ E[u_A] \)
- The local incentive constraint writes now as
  \[ \frac{\partial E[u_A]}{\partial e} = \sum_{m=1}^{M} p'_m(e)w_m - c'(e) = 0 \quad (\text{IC}') \]
- The participation constraint is
  \[ E[u_A] \geq u_A^{\text{res}} \iff \sum_{m=1}^{M} p_m(e)w_m - c(e) \geq u_A^{\text{res}} \quad (\text{PC}) \]
Solution

- Setting incentives – and thereby aligning interests – is still costless for the principal.
- It is thus optimal for the principal to perfectly align the interests by making the agent the residual claimant (i.e., “selling the firm”):
  \[ w_m^* = r_m - F, \]
  where \( F \) is a fixed amount which is such that the (PC) is binding, i.e.,
  \[ F = \sum_{m=1}^{M} p_m(e^*) r_m - c(e^*) - u_A^{res} \]
- Therefore, the results stated in Proposition 1 are valid also with \( M > 2 \) outcomes.
- That is, despite effort being not contractible, the principal
  - implements the first-best effort (=efficient effort) level,
  - only has to bear the first-best expected wage costs,
  - and experiences the first-best expected utility.
Remark

- In contrast to the case with just two possible outcomes, multiple contracts are now optimal.
- For example, instead of “selling the firm”, the principal can also pick one outcome with $p'_m(e^{FB}) > 0$, say $\dot{m}$, and set

$$w_m = \begin{cases} 
(1 - p_m(e^{FB})) \frac{c'(e^{FB})}{p'_m(e^{FB})} + c(e^{FB}) + u_A^{res} & \text{for } m = \dot{m}, \\
-p_m(e^{FB}) \frac{c'(e^{FB})}{p'_m(e^{FB})} + c(e^{FB}) + u_A^{res} & \text{otherwise}
\end{cases}$$
3.3 Limited Liability
Motivation and preview of results

- With unlimited liability, the principal is able to set arbitrarily low wages, in particular, negative wages.
- In the real-world, this is often not possible due to laws protecting agents (e.g., minimum wages) or the agents’ limited wealth.
- It is thus interesting to explore the scenario where wages cannot be arbitrarily low.
- We then say that the agent is protected by limited liability or equivalently that he is wealth constrained.

Preview of Results:

- The agent’s participation constraint does not bind.
- The agent’s expected utility is increasing in the implemented effort.
- The principal implements an effort level that is lower than the first-best effort and the efficient effort: \( e^* < e^{FB} = e^{effi} \).
Formalization

Due to limited liability, wages cannot fall short of a certain threshold:

\[ w_m \geq w^{\text{threshold}} \text{ for all } m \in \{1, \ldots, M\} \]

It is convenient to normalize \( w^{\text{threshold}} = 0 \)

- A normalization is an adjustment of the units in how we measure some quantity
- Normalizations are without loss of generality, i.e., do not change the results
- For example, if hourly wages cannot fall short of 9,19 Euro, then we can normalize \( w \) such that it measures the hourly wage exceeding 9,19 Euro
Claim 1

If the agent’s reservation utility $u_{A}^{\text{res}}$ is sufficiently high (in case $M = 2$, $u_{A}^{\text{res}} \geq p(e^{FB})\Delta r - c(e^{FB})$), the limited liability constraints have no effect.

Proof for the case $M = 2$:

- In the optimum without limited liability, see Chapter 3.2,
  \[ w_2^* > w_1^* = -p(e^*)\Delta r + c(e^*) + u_{A}^{\text{res}} = -p(e^{FB})\Delta r + c(e^{FB}) + u_{A}^{\text{res}} \]

- Therefore, if
  \[ -p(e^{FB})\Delta r + c(e^{FB}) + u_{A}^{\text{res}} \geq w^{\text{threshold}} = 0 \]

  \[ \iff u_{A}^{\text{res}} \geq p(e^{FB})\Delta r - c(e^{FB}), \]

the limited liability constraints $w_1, w_2 \geq 0$ can be neglected in the principal’s maximization problem and have therefore no effect \[\square\]
Reservation utility

- We henceforth set $u_A^{res} = 0$

**Claim 2**

If $u_A^{res} = 0$, then the agent's participation constraint (PC) is automatically satisfied and can hence be neglected.

Proof:

1. Since the agent chooses the effort to maximize his expected utility, $E[u_A]|_{e^*} \geq E[u_A]|_{e=0}$, where $e^*$ denotes the agent's optimal effort choice given some contract $(w_1, w_2)$

2. Due to limited liability and $c(0) = 0$ we have

$$E[u_A]|_{e=0} = \sum_{m=1}^{M} p_m(0)w_m - c(0) = \sum_{m=1}^{M} p_m(0)w_m \geq 0$$

3. Hence, $E[u_A]|_{e^*} \geq u_A^{res} = 0$
Benchmarks

- Before we analyze the principal’s problem, it is important to recognize that both benchmarks are unaffected by limited liability.
- In benchmark I (welfare maximum), the wage does not matter such that we can pick some wage that satisfies limited liability and the efficient effort $e^\text{effi}$ still equalizes the marginal return of effort and the marginal costs of effort.
- In benchmark II (contractible effort), the principal can still implement $e^{FB} = e^\text{effi}$ for the wage $w^{FB} = c(e^{FB}) + u^\text{res}_A$, since for $u^\text{res}_A = 0$ we have $w^{FB} = c(e^{FB}) > w^\text{threshold} = 0$. 
The principal’s problem

- We go back to the case with non-contractible effort and let $M = 2$
- The principal designs the contract to maximize her expected utility subject to the incentive constraint, the participation constraint, and the limited liability constraints
- From before we know that for $\Delta w \geq 0$ we can use the first-order approach, i.e., replace the global incentive constraint (IC) by the local incentive constraint (IC’)
- Moreover, $\Delta w \geq 0$, which is equivalent to $w_2 \geq w_1$, and $w_1 \geq 0$ implies that $w_2 \geq 0$
- We are again confident that $\Delta w \geq 0$ and thus solve the principal's relaxed problem:

$$\max_{(w_1, w_2)} E[u_P] \text{ subject to (IC’), } w_1 \geq 0$$
Assumption

- $\Delta r$ is sufficiently high:
  \[
  \Delta r > \frac{p(0)}{(p'(0))^2} c''(0)
  \]

- Interpretation: yielding outcome 2 instead of 1 (i.e., the successful outcome instead of the unsuccessful one) is sufficiently important for the principal.

- Note that the assumption is always satisfied – i.e., also for arbitrarily low values of $\Delta r$ – if effort is essential in the sense that $p(0) = 0$.

- As we will see below, the assumption guarantees that implementing zero effort is never optimal for the principal.
The Lagrangian writes

\[ \mathcal{L}(e, w_1, w_2) = p(e)(r_2 - w_2) + (1 - p(e))(r_1 - w_1) \]
\[ + \mu(p'(e)(w_2 - w_1) - c'(e)) \]
\[ + \rho(w_1 - 0) \]
Optimum

- In the optimum, the following holds:

\[
\frac{\partial \mathcal{L}}{\partial w_1}(e, w_1, w_2) = -1 + p(e) - \mu p'(e) + \rho = 0 \quad (8)
\]

\[
\frac{\partial \mathcal{L}}{\partial w_2}(e, w_1, w_2) = -p(e) + \mu p'(e) = 0 \quad (9)
\]

\[
\frac{\partial \mathcal{L}}{\partial e}(e, w_1, w_2) = p'(e)(\Delta r - \Delta w) + \mu (p''(e)\Delta w - c''(e)) = 0 \quad (10)
\]

\[
p'(e)(w_2 - w_1) = c'(e) \quad (IC')
\]

\[
w_1 \geq 0 \quad (LL)
\]

\[
\rho \geq 0; \text{ if } w_1 > 0, \text{ then } \rho = 0 \quad (11)
\]
Solution

- Adding (8) and (9) yields $\rho^* = 1$
- Therefore, the limited liability constraint $w_1 \geq 0$ has to bind, which implies that $w_1^* = 0$
- Plugging $\rho^* = 1$ in (9) (or (8)) yields that

$$\mu^* = \frac{p(e)}{p'(e)}$$

- Plugging this into (10) we get

$$p'(e)(\Delta r - \Delta w) + \frac{p(e)}{p'(e)}(p''(e)\Delta w - c''(e)) = 0 \quad (12)$$
Due to the assumption we made before, $\Delta w = 0$ cannot be a solution of (12); the left-hand side is then positive.

And since the left-hand side is decreasing in $\Delta w$, the left-hand side of (12) is positive also if $\Delta w < 0$.

If $\Delta w = \Delta r$, the left-hand side of (12) is negative.

And this also holds for $\Delta w > \Delta r$.

Since all terms are continuous in $\Delta w$, there must exists an optimum $\Delta w^*$ and in the optimum we must have $\Delta w^* \in (0, \Delta r)$.

Recall that for $\Delta w = 0$ the agent chooses $e = 0$, for $\Delta w = \Delta r$ the agent chooses $e = e^{effi}$, and $e$ is increasing in $\Delta w$, the optimally implemented effort $e^*$ is hence strictly between 0 and $e^{FB} = e^{effi}$.

Formally, the optimal values $\Delta w^* = w_2^*$ and $e^*$ solve (12) and ($IC'$).
Agent’s rent

- The agent’s expected utility in excess of his reservation utility is denoted as the agent’s rent.
- In case of unlimited liability, the agent’s rent is zero for the cost-minimizing contract, no matter what effort level the principal implements.
  - Reason: (PC) is binding for all effort levels, i.e., $E[u_A] = u_A^{res}$
  - Intuition: If the agent would experience a positive rent, such that $E[u_A] > u_A^{res}$, then the principal can lower her wage costs and thereby improve her expected utility by lowering all wages until $E[u_A] = u_A^{res}$
Agent’s rent with limited liability

- If the principal implements effort $\hat{e} = 0$, the cost-minimizing (and thus optimal) contract is $w_1 = w_2 = 0$
- Then $E[u_A] = p(e)w_2 + (1 - p(e))w_1 - c(e) = 0$ and so the agent’s rent is zero
- If the principal wants to implement a positive effort $\hat{e}$, she must increase $w_2$ to the level where $w_2 = c'(\hat{e})/p'(\hat{e})$
- The agent’s expected utility is
  $$E[u_A] = p(e)w_2 + (1 - p(e))w_1 - c(e)$$
- Having a positive wage $w_2$ and a positive effort level leads to a positive expected wage payment, which increases $E[u_A]$, ...
- ... but also to positive effort costs, which decreases $E[u_A]$
- How is the agent’s expected utility affected?
By the Envelope Theorem
\[
\frac{dE[u_A]}{dw_2} = \frac{\partial E[u_A]}{\partial w_2} + \frac{\partial E[u_A]}{\partial e} \cdot \frac{\partial e}{\partial w_2} = \frac{\partial E[u_A]}{\partial w_2} = p(e) > 0
\]

Hence, the agent’s expected utility and thereby also his rent is increasing in \( w_2 \), i.e., increasing in \( \hat{e} \)

Thus, the principal must pay the agent a rent for any \( \hat{e} > 0 \) and the rent is increasing in \( \hat{e} \)

Therefore, setting incentives – and thereby aligning interests – is costly for the principal!
Principal’s trade-off: rent extraction vs efficiency

- With limited liability, the principal faces the following trade-off:
  1. Rent extraction: implement a low effort level to extract some of the rent she has to pay to the agent (to minimize the rent, \( \hat{e} = 0 \) and so \( \Delta w = 0 \) is optimal)
  2. Efficiency: implement a high effort level – i.e., the efficient effort level – to maximize efficiency (to maximize efficiency, \( \hat{e} = e^{effi} \) and so \( \Delta w = \Delta r \) is optimal)

- As we have seen above, it is optimal for the principal to implement an intermediate effort level \( e^* \in (0, e^{effi}) \) by setting \( \Delta w^* = w_2^* \) strictly between 0 and \( \Delta r \)

- Remark: the principal could implement the efficient effort level by setting \( w_1^* = 0 \) and \( w_2^* = \Delta r \); but this is not optimal for the principal, due to the high rent she would then have to pay to the agent
We next want to explore more precisely why the results with unlimited and with limited liability are different.

Recall that the expected welfare is $E[V] = E[u_P] + E[u_A]$.

The principal maximizes $E[u_P]$.

This is equivalent to maximizing $E[V] - E[u_A]$. 

Intuition: unlimited versus limited liability
Unlimited liability

- With unlimited liability the agent receives no rent, no matter what effort the principal implements.
- That is, $E[u_A] = u_A^{\text{res}}$
- Maximizing $E[u_P] = E[V] - E[u_A] = E[V] - u_A^{\text{res}}$ over effort yields the same optimal effort level as maximizing $E[V]$.
- The principal hence implements the efficient effort, $e^* = e^{\text{effi}}$.
- There is no rent-efficiency trade-off.
Limited Liability

- Again, the principal maximizes $E[u_P] = E[V] - E[u_A]$
- Start with the implemented effort level $e = 0$
- From before we know that increasing the implement effort level increases $E[V]$ as well as the agent’s rent $E[u_A] - u_{res}^A$ and thus also $E[u_A]$.
- The principal does not increase the implemented effort level until $E[V]$ is maximal, but only until $E[u_P] = E[V] - E[u_A]$ is maximal.
- Thus, with limited liability the principal implements less than the efficient effort, $e^* < e^{effi}$.
- There is a rent-efficiency trade-off.
Illustration

- Consider the following example: \( p(e) = e, \ c(e) = e^2 / 2, \ r_1 = 0, \ r_2 = 1, \) and \( u_A^{res} = 0 \)

- Suppose first that the agent is protected by limited liability

- In the optimum \( w_1^* = 0 \)

- From Exercises 3.3 and 3.4 we know that then
  
  - \( e = w_2 \)
  
  - \( E[u_P] = w_2(1 - w_2) \)
  
  - \( E[u_A] = \frac{w_2^2}{2} \)
  
  - Thus, \( E[V] = w_2 - \frac{w_2^2}{2} \)
Limited liability

\[ E[V] \]

\[ E[u_A] \]

\[ E[u_P] \]
Implementation of efficient effort with lim. liability

- To implement the efficient effort $e^{effi} = 1$, the principal would need to set $\Delta w = \Delta r = 1$
- The optimal contract to implement $\hat{e} = 1$ is hence $(w_1 = 0, w_2 = 1)$
- But then $w_1 = r_1$ and $w_2 = r_2$
- Hence $u_P = 0$ in case of outcome 1 and in case of outcome 2 and so $E[u_P] = 0$
- To give away the whole return to the agent is obviously not optimal, although this would implement $e^{effi}$ and therefore maximize the expected welfare $E[V]$
- To maximize her expected utility $E[u_P]$, the principal optimally only implements effort $\hat{e} = 1/2$ by the contract $(w_1^* = 0, w_2^* = 1/2)$
- This yields her an expected utility of $E[u_P] = 1/4$
Unlimited liability
To implement the efficient effort $e^{effi} = 1$, the principal still needs to set $\Delta w = \Delta r = 1$.

But with unlimited liability the optimal contract to implement $\hat{e} = 1$ is $(w_1^* = -1/2, w_2^* = 1/2)$.

The agent’s participation constraint then just binds.

Since the principal can now set negative wages, it is indeed optimal for her to implement $\hat{e} = 1$ by this contract.

This yields her $E[u_P] = 1/2$. 
The agent benefits from limited liability since $E[u_A]|_{\text{lim. lia.}} > u_A^{\text{res}}$, while $E[u_A]|_{\text{unlim. lia.}} = u_A^{\text{res}}$, and so

$$E[u_A]|_{\text{lim. lia.}} > E[u_A]|_{\text{unlim. lia.}}$$

The principal is harmed by limited limited liability:

$$E[u_P]|_{\text{lim. lia.}} = E[V]|_{e^{*}} \neq e^{\text{effi}} - E[u_A]|_{\text{lim. lia.}}$$

$$< E[V]|_{e^{*}} = e^{\text{effi}} - E[u_A]|_{\text{unlim. lia.}}$$

$$= E[u_P]|_{\text{unlim. lia.}}$$
Interpretation

- Non-contractibility of effort is bad for the principal (and also welfare) in case of limited liability even though in equilibrium the principal knows exactly what effort the agent is choosing.

- The principal designs a contract that ensures that the agent is choosing a particular level of effort.

- The contract thus has to ensure it will be desirable for the agent to choose the particular effort level and this is costly for the principal.
**Proposition 2**

Suppose effort is not contractible, the agent is risk neutral, the agent’s liability is limited \((w_m \geq 0 \text{ for all } m \in \{1, \ldots, M\})\), and the agent’s reservation utility is \(u^{\text{res}}_A = 0\). The principal implements a positive, but inefficiently low effort level: \(0 < e^* < e^{FB} = e^{effi}\). The principal optimally sets a contract \((w_1^* = 0, w_2^*)\), where \(w_2^* \in (0, \Delta r)\). The principal’s expected utility is lower than in case of contractible effort (or non-contractible effort and unlimited liability), while the agent’s expected utility is higher than in case of contractible effort (or non-contractible effort and unlimited liability). The participation constraint does not bind and the agent earns a positive rent \(E[u_A] > u^{\text{res}}_A = 0\).
3.3.1 Limited Liability and $M > 2$

Outcomes
Suppose there are $M > 2$ possible outcomes and that the principal seeks to implement a positive effort level, so $\hat{e} > 0$

The agent maximizes

$$E[u_A] = \sum_{m=1}^{M} p_m(e)w_m - c(e)$$

The incentive constraint is

$$\hat{e} \in \text{argmax } E[u_A]$$

The first-order condition of the agent’s maximization problem is

$$\frac{\partial E[u_A]}{\partial e} = \sum_{m=1}^{M} p'_m(e)w_m - c'(e) = 0$$

We take as given that the first-order approach is valid
Principal’s problem

- The principal problem is to
  \[
  \max_{(w_1, \ldots, w_M)} E[u_P] \text{ subject to } (PC), (IC'), \ w_1, \ldots, w_M \geq 0
  \]

- As in case of \( M = 2 \) outcomes, the participation constraint is satisfied automatically due to limited liability and \( u_A^{res} = 0 \)

- The Lagrangian writes
  \[
  \mathcal{L}(e, w_1, \ldots, w_M) = \sum_{m=1}^{M} p_m(e)(r_m - w_m) \\
  + \mu \left( \sum_{m=1}^{M} p_m'(e)w_m - c'(e) \right) \\
  + \sum_{m=1}^{M} \rho_m(w_m - 0)
  \]
Optimum

- In the optimum, the following holds:

\[
\frac{\partial \mathcal{L}(e, w_1, \ldots, w_M)}{\partial w_m} = -p_m(e) + \mu p'_m(e) + \rho_m = 0 \quad \forall m \in \{1, \ldots, M\} \tag{13}
\]

\[
\frac{\partial \mathcal{L}(e, w_1, \ldots, w_M)}{\partial e} = \sum_{m=1}^{M} p'_m(e)(r_m - w_m) + \mu \left( \sum_{m=1}^{M} p''_m(e)w_m - c''(e) \right) \tag{14}
\]

\[
\sum_{m=1}^{M} p'_m(e)w_m = c'(e) \tag{IC'}
\]

\[
\rho_m \geq 0; \text{ if } w_m > 0, \text{ then } \rho_m = 0 \quad \forall m \in \{1, \ldots, M\} \tag{15}
\]
Solution

- By (13), if $\mu^* = 0$, then $\rho_m = p_m(e)$ and so $w_m = 0$ for all $m \in \{1, \ldots, M\}$ and so $\sum_{m=1}^{M} p'_m(e) w_m = 0$, which violates (IC').

- By (13), if $\mu^* < 0$, then it must hold that $\rho_m > 0$ for all outcomes with $p'_m(e) \geq 0$ and so $w_m = 0$ for all these outcomes, which again violates (IC') since then $\sum_{m=1}^{M} p'_m(e) w_m \leq 0$.

Therefore, $\mu^* > 0$ must be true.

- From (13) we see that for all outcomes with $p'_m(e) \leq 0$ it must hold that $\rho_m > 0$ and so that $w^*_m = 0$.

This is intuitive: Since the principal wants to convince the agent to invest effort by setting monetary incentives, it would be counterproductive to reward the agent for outcomes which more likely arise if the agent invests less effort.
Thus, the wage \( w_m \) could only be positive for outcomes \( m \) with 
\[ p'_m(e) > 0 \]

For which of these outcomes should \( w_m > 0 \)?

**Claim 3**

*In the optimal contract, wages can only be positive for the outcome(s) with the maximal likelihood ratio \( p'_m(e)/p_m(e) \).*

**Proof:**

Suppose, contrary to the claim, that there are outcomes \( k \) and \( l \) with

\[ p'_k(e)/p_k(e) < p'_l(e)/p_l(e) \]

and \( w_k > 0 \)
Dividing (13) by $p_m(e)$ we get

$$-1 + \frac{\rho_m}{p_m(e)} + \mu \frac{p'_m(e)}{p_m(e)} = 0$$

Since $w_k > 0$, we have $\rho_k = 0$ and so

$$-1 + \mu \frac{p'_k(e)}{p_k(e)} = 0$$

But then, since $\rho_l \geq 0$ and $\mu > 0$,

$$-1 + \frac{\rho_l}{p_l(e)} + \mu \frac{p'_l(e)}{p_l(e)} = 0$$

cannot hold □
For simplicity, we suppose that there is only one outcome that possesses the highest likelihood ratio.

Denote this outcome as \( \tilde{m} \)

Then \( w^*_m = 0 \) for all \( m \in \{1, \ldots, M\} \setminus \tilde{m} \)

\( w_{\tilde{m}} \) is then determined by (IC’):

\[
w^*_{\tilde{m}} = \frac{c'(e)}{p'_{\tilde{m}}(e)}
\]

The optimal contract takes a very simple form:

- The agent always receives a base wage that equals the limited liability threshold \( w^{\text{threshold}} \) (which we have normalized to zero) and ...
- ... a bonus of \( w^*_{\tilde{m}} - w^{\text{threshold}} \) in case the outcome is \( \tilde{m} \)
The $M > 2$-outcome problem hence collapses to the two-outcome problem:

- The outcome is either $\tilde{m}$, in which case the agent is remunerated by a positive wage, or ...
- ... different, in which case the agent receives a wage of zero

The remaining problem of determining the optimal effort (see the above Lagrangian) is analogous to the two-outcome problem.

Therefore, except that outcome $\tilde{m}$ is rewarded (i.e., not necessarily outcome 2), Proposition 2 stays valid also with $M > 2$ possible outcomes.
Outcomes and wages

- In reality, a bonus is often paid if the return exceeds a certain level.
- Does the theoretical optimal contract we specified have this property?
- Not necessarily, since $\tilde{m}$ need not coincide with the outcome where the return is the highest, namely outcome $M$.
- The optimal contract has this property if and only if

$$p'_M(e)/p_M(e) > p'_m(e)/p_m(e)$$

for all $m \in \{1, \ldots, M - 1\}$ since then $\tilde{m} = M$. 
Intuition

- Why is it optimal for the principal only to remunerate the agent for the outcome with the maximal likelihood ratio?
- The agent is rewarded in the state of nature which is the most informative one about the fact that he has exerted a certain effort level (for more details on this intuition, see Laffont and Martimort Ch. 3.6)

- Alternative interpretation (my interpretation):
  - $p_m(e)$ is the principal’s marginal cost of increasing $w_m$
  - $p'_m(e)$ is the marginal incentive effect of increasing $w_m$
  - The outcome with the highest likelihood ratio offers the best “bang for the buck”, i.e., the highest incentive effect relative to the costs
Remark

- In case multiple outcomes have the maximal likelihood ratio, it is still optimal not to reward the outcomes that have not the highest likelihood ratio.

- However, there are then multiple optimal contracts, since the principal is indifferent which of the outcomes (or which combination of outcomes) with the maximal likelihood ratio to remunerate.
Binary effort

- Suppose effort is binary: \( e \in \{0, 1\} \)

- With continuous effort (see above) the likelihood ratio of outcome \( m \) is \( \frac{p_m'(e)}{p_m(e)} \)

- With binary effort the likelihood ratio of outcome \( m \) is \( \frac{p_m(1) - p_m(0)}{p_m(1)} \)

- We still get the result that it is optimal only to reward the outcome with the maximal likelihood ratio

Remark:

- Some textbooks use \( \frac{p_m(1)}{p_m(0)} \) for the likelihood ration, instead of \( \frac{p_m(1) - p_m(0)}{p_m(1)} \)
- No matter which of the two definitions we use, the ordering of likelihood ratios is the same
To see this, consider the likelihood ratios of two outcomes, \( l \) and \( k \):

\[
\frac{p_l(1) - p_l(0)}{p_l(1)} > \frac{p_k(1) - p_k(0)}{p_k(1)}
\]

\[
\iff 1 - \frac{p_l(0)}{p_l(1)} > 1 - \frac{p_k(0)}{p_k(1)}
\]

\[
\iff \frac{p_l(0)}{p_l(1)} < \frac{p_k(0)}{p_k(1)}
\]

\[
\iff \frac{p_l(1)}{p_l(0)} > \frac{p_k(1)}{p_k(0)}
\]

Thus, if \( \frac{p_l(1) - p_l(0)}{p_l(1)} > \frac{p_k(1) - p_k(0)}{p_k(1)} \) for two outcomes \( l \) and \( k \), then also \( \frac{p_l(1)}{p_l(0)} > \frac{p_k(1)}{p_k(0)} \) and vice versa.
3.4 Risk Aversion
Motivation

- We next suppose that the agent is risk averse
- Definition: An agent is risk averse if for all non-degenerated random variables $X$ it holds that $u(E[X]) > E[u(X)]$
- That is, the agent prefers to receive the expected value of the random variable for sure instead of the random variable itself
- This is very plausible, since many persons are averse towards risks
- For example, we throw a coin: you receive 0 Euro if the coin shows heads and 200 Euro if it shows tails; do you prefer to receive 100 Euro for sure instead of this random variable?
Agent’s utility

- Let the agent’s utility be \( u_A = u(w) - c(e) \), where \( u : \mathbb{R} \to \mathbb{R} \) and \( u \) is twice continuously differentiable.

- The agent’s expected utility is thus

\[
E[u_A] = \sum_{m=1}^{M} p_m(e) u(w_m) - c(e)
\]

- We suppose that \( u' > 0 \) (so that the marginal utility of income is positive) and that \( u'' < 0 \) (so that the agent is risk averse).

- Furthermore, we suppose that \( \lim_{w \to \infty} u_A = \infty \)

  - To see purpose of this assumption, suppose that it is violated. For example, let \( \lim_{w \to \infty} u_A = 10 \). If the agent’s reservation utility is \( u_A^{res} = 12 \), then the principal cannot design a contract that satisfies the agent’s participation constraint.

  - We could alternatively assume that \( \lim_{w \to \infty} u_A \) is “sufficiently large”
Preliminary notes

- For simplicity, we suppose that the first-order approach is valid.
- This requires some technical conditions we do not want to discuss.
- The analysis will be relatively brief and ignore some technical details.
- There is unlimited liability (the case where the agent is risk averse and also protected by limited liability is very complicated).

Claim 4

If the principal wants to implement \( \hat{e} = 0 \), then the optimal contract is to pay \( w = \hat{w} \) for all outcomes, i.e., \( w_m = \hat{w} \) for all \( m \in \{1, \ldots, M\} \), where \( \hat{w} \) solves \( u(w) = u_A^{res} \).
Proof:

- We solve the relaxed problem without an incentive constraint and then verify that it is satisfied.
- The principal’s problem is to

\[
\max_{(w_1, \ldots, w_M)} E[u_P] \text{ subject to (PC)}
\]

- The Lagrangian writes

\[
\mathcal{L}(\cdot) = \sum_{m=1}^{M} p_m(e)(r_m - w_m) + \lambda \left( \sum_{m=1}^{M} p_m(e)u(w_m) - c(e) - u_A^{res} \right)
\]

- Differentiating yields

\[
\frac{\partial \mathcal{L}(\cdot)}{\partial w_m} = -p_m(e) + \lambda p_m(e)u'(w_m) = 0
\]

- Hence, for all \( m \in \{1, \ldots, M\} \) it must hold that

\[
\lambda u'(w_m) = 1
\]

- Thus, the wage \( w_m \) must be the same for all \( m \in \{1, \ldots, M\} \), say \( w_m = \hat{w} \).
Because $\lambda^* > 0$, see the equation before, the participation constraint (PC) must bind such that

$$
\sum_{m=1}^{M} p_m(e) u(w_m) - c(0) = u_A^{res}
$$

$$
\iff \sum_{m=1}^{M} p_m(e) u(\dot{w}) = u_A^{res}
$$

$$
\iff u(\dot{w}) \sum_{m=1}^{M} p_m(e) = u_A^{res}
$$

$$
\iff u(\dot{w}) = u_A^{res}
$$

It remains to check that the incentive constraint (IC) is satisfied.

Since the wage is fixed, $w_m = \dot{w}$ for all $m \in \{1, \ldots, M\}$, the agent optimally chooses the lowest possible effort level $e = 0$. \qed
Agent’s problem

- We henceforth suppose that the principal seeks to implement a positive effort level (so $\hat{e} > 0$), but do not solve for the effort she optimally implements.
- Having accepted the contract suggested by the principal, the agent maximizes his expected utility over his effort choice.
- The incentive constraint is thus
  $$\hat{e} \in \arg\max E[u_A]$$  
  (IC)
- The first-order condition of the agent’s problem is
  $$\frac{\partial E[u_A]}{\partial e} = \sum_{m=1}^{M} p'_m(e)u(w_m) - c'(e) = 0$$  
  (IC')
The agent optimally participates, i.e., accepts the contract suggested by the principal, if and only if

\[ E[u_A] = \sum_{m=1}^{M} p_m(e)u(w_m) - c(e) \geq u_A^{res}, \]  

(PC)

where effort \( e \) is optimally chosen.

The participation constraint is a condition about absolute utilities \((u(w_1), \ldots, u(w_M))\). The incentive constraints (IC') and (IC) are conditions about relative utilities \((u(w_1) - u(w_M), \ldots, u(w_M) - u(w_M))\).

To see that (IC') is a condition about relative utilities, note that \( \sum_{m=1}^{M} p_m(e) = 1 \) for all efforts \( e \) such that \( \sum_{m=1}^{M} p'_m(e) = 0 \).
We can hence rewrite (IC'):

\[
\sum_{m=1}^{M} p'_m(e)u(w_m) - c'(e) = 0
\]

\[\iff \sum_{m=1}^{M} p'_m(e)u(w_m) - u(w_M)\sum_{m=1}^{M} p'_m(e) - c'(e) = 0\]

\[\iff \sum_{m=1}^{M} p'_m(e)(u(w_m) - u(w_M)) - c'(e) = 0\]
Principal’s problem

- The principal maximizes her expected utility subject to the agent’s participation constraint and incentive constraint
- Applying the first-order approach, her problem is to
  \[
  \max_{(w_1, \ldots, w_M)} E[u_P] \text{ subject to } (PC), (IC')
  \]
- The Lagrangian writes

\[
\mathcal{L}(e, w_1, \ldots, w_M) = \sum_{m=1}^{M} p_m(e)(r_m - w_m) \\
+ \lambda \left( \sum_{m=1}^{M} p_m(e)u(w_m) - c(e) - u_A^{res} \right) \\
+ \mu \left( \sum_{m=1}^{M} p'_m(e)u(w_m) - c'(e) \right)
\]
Differentiating yields

\[
\frac{\partial \mathcal{L}(e, w_1, \ldots, w_M)}{\partial w_m} = -p_m(e) + \lambda p_m(e)u'(w_m) + \mu p'_m(e)u'(w_m) = 0,
\]

which must hold for all \( m \in \{1, \ldots, M\} \)

For the interpretation of this formula it is important to determine the signs of \( \lambda^* \) and \( \mu^* \)

Since (IC') is an equality constraint, \( \mu^* \) may take any sign

Since (PC) is an inequality constraint, we have that \( \lambda^* \geq 0 \)

And if \( \sum_{m=1}^M p_m(e)u(w_m) - c(e) - u_{A}^{res} > 0 \), then \( \lambda^* = 0 \)
Determining the sign of $\lambda^*$

**Claim 5**

The Lagrange parameter of the participation constraint (PC) is positive: $\lambda^* > 0$.

**Proof:**

- If $\lambda^* = 0$, (16) simplifies to
  \[-p_m(e) + \mu p'_m(e)u'(w_m) = 0\]  \hspace{1cm} (17)

- If additionally $\mu^* = 0$, then $-p_m(e) = 0$, which cannot hold.

- If additionally $\mu^* > [\leq]0$, (17) cannot be satisfied for an outcome with $p'_m(e) \leq [\geq]0$.

- Note that such an outcome has to exists since $\sum_{m=1}^{M} p_m(e) = 1$ has to hold for all $e$ and so $\sum_{m=1}^{M} p'_m(e) = 0$.

- Thus, $\lambda^* > 0$
Intuition

• Suppose the principal chooses an optimal contract
• We then lower the agent’s reservation utility $u_A^{res}$
• The principal can then lower all wages such that all utilities $u(w_1), \ldots, u(w_M)$ decrease by $\varepsilon$, where $\varepsilon$ is small and positive
• This does not change the agent’s incentives …
• … but reduces the principal’s expected wage payment
• Thus, the principal’s expected utility improves
• Note that $\frac{\partial L(\cdot)}{\partial (-u_A^{res})} = \lambda$, i.e., the Lagrange parameter $\lambda$ shows how much the maximal value of the objective function $E[u_P]$ changes if we increase $-u_A^{res}$, i.e., decrease $u_A^{res}$, by one (marginal) unit
• Hence, $\lambda^*$ must be positive
Determining the sign of $\mu^*$

**Claim 6**

The Lagrange parameter of the local incentive constraint (IC’) is positive: $\mu^* > 0$.

Proof:

- If $\mu^* = 0$, (16) simplifies to
  
  $$-p_m(e) + \lambda p_m(e)u'(w_m) = 0 \iff -1 + \lambda u'(w_m) = 0$$

  and implies that $w_m$ is constant for all $m \in \{1, \ldots, M\}$

- But then (IC’) is violated, since
  
  $$\sum_{m=1}^{M} p'_m(e)u(w_m) = u(w_m)\sum_{m=1}^{M} p'_m(e) = 0 \neq c'(e)$$

- If $\mu^* < 0$, (16) can be written as
  
  $$\frac{1}{u'(w_m)} + |\mu| \frac{p'_m(e)}{p_m(e)} = \lambda$$

  (18)

- Implicit differentiation yields that $\frac{\partial w_m}{\partial (p'_m(\cdot)/p_m(\cdot))} < 0$
Let \( \tilde{w} \) solve \( \frac{1}{u'(w_m)} + |\mu| \frac{0}{p_m(e)} = \lambda \) and define \( \tilde{u} := u(\tilde{w}) \).

Then, according to (18), \( u(w_m) > \tilde{u} \) if \( p'_m(\cdot) < 0 \) and \( u(w_m) < \tilde{u} \) if \( p'_m(\cdot) > 0 \).

Recall that
\[
\sum_{m=1}^{M} p'_m(e) u(w_m) - c'(e) = 0 \quad \text{(IC')}\]

Since \( \sum_{m=1}^{M} p'_m(e) = 0 \) we also have that
\[
\tilde{u} \sum_{m=1}^{M} p'_m(e) = \sum_{m=1}^{M} p'_m(e) \tilde{u} = 0
\]

Thus
\[
\sum_{m=1}^{M} p'_m(e) u(w_m) = \sum_{m=1}^{M} p'_m(e) (u(w_m) - \tilde{u}) < 0
\]

But then (IC') is violated

Hence, \( \mu^* > 0 \)
Intuition

- Suppose the principal chooses an optimal contract
- We then lower the agent’s marginal costs $c'(e)$
- The principal can then offer a contract with lower wage spreads, which lowers the risk premium she has to pay to the agent, and thereby increases the principal’s expected utility
- Note that $\frac{\partial L(\cdot)}{\partial(-c'(e))} = \mu$, i.e., the Lagrange parameter $\mu$ shows how much the maximal value of the objective function $E[u_P]$ changes if we increase $-c'(e)$, i.e., decrease $c'(e)$, by one (marginal) unit
- Hence, $\mu^*$ must be positive
Holmström formula

- Rewriting (16) yields the following important formula:

\[
\frac{1}{u'(w_m)} = \lambda + \mu \frac{p'_m(e)}{p_m(e)}
\]  

(HF)

for all \( m \in \{1, \ldots, M\} \)

- We denote this formula as the Holmström formula, since it is due to Holmström (1979)

- Because \( \lambda^*, \mu^* > 0 \), the Holmström formula implies that the wage \( w_m \) is higher the higher is the likelihood ratio \( \frac{p'_m(e)}{p_m(e)} \) of outcome \( m \)

- Thus, as in the case with risk neutrality and limited liability, higher outcomes are not generally remunerated more highly

- If the Monotone Likelihood Ratio Property holds, \( \frac{p'_m(e)}{p_m(e)} \) is increasing in \( m \), then higher outcomes (i.e., those with a higher index \( m \)) are remunerated more highly
**Proposition 3**

Suppose effort is not contractible, the agent is risk averse ($u' > 0$ and $u'' < 0$), and the agent’s liability is unlimited. To implement effort $\hat{e} > 0$, the principal optimally sets a contract $(w_1^*, \ldots, w_M^*)$ that satisfies the Holmström formula:

$$\frac{1}{u'(w_m)} = \lambda + \mu \frac{p'_m(e)}{p_m(e)},$$

(HF)

where the Lagrange multipliers are positive, $\lambda^*, \mu^* > 0$. An outcome $m$ with a higher likelihood ratio $\frac{p'_m(e)}{p_m(e)}$ is rewarded with a higher wage $w_m^*$. If the Monotone Likelihood Ratio Property holds, such that $\frac{p'_m(e)}{p_m(e)}$ is increasing in $m$, then $w_{m+1}^* > w_m^*$ for all $m \in \{1, \ldots, M - 1\}$. 
Risk premium

- Suppose \( w_m \) is not constant for all \( m \in \{1, \ldots, M\} \)
- The risk averse agent prefers to receive the expected wage payment \( \sum_{m=1}^{M} p_m(e)w_m \) for sure instead of the lottery induced by the original contract:

\[
u \left( \sum_{m=1}^{M} p_m(e)w_m \right) > \sum_{m=1}^{M} p_m(e)u(w_m)\]

- The certainty equivalent solves

\[
u(CE) = \sum_{m=1}^{M} p_m(e)u(w_m)\]

- Since \( u' > 0 \) we have \( CE < \sum_{m=1}^{M} p_m(e)w_m \)
- The risk premium is \( RP := \sum_{m=1}^{M} p_m(e)w_m - CE > 0 \)
Principal’s costs

- Note that because $\lambda^* > 0$, the agent’s participation constraint (PC) binds.
- The principal thus has to compensate the agent for
  - (i) his outside option (i.e., pay his reservation utility),
  - (ii) his effort costs, and
  - (iii) the risk he faces (i.e., pay him a risk premium)
- Since the principal must pay the agent a risk premium, setting incentives – and thereby aligning interests – is costly for the principal!
The principal faces the following trade-off:

1. **Insurance:** To optimally insure the agent and minimize the agent’s risk premium the principal has to pay, it is optimal to set a fixed wage, i.e., provide no incentives, and thereby implement zero effort.

2. **Efficiency:** To maximize efficiency it is optimal to make the agent the residual claimant, i.e., to provide a lot of incentives, and thereby implement the efficient effort.
Comparison

- Due to this trade-off, the principal optimally implements less effort than when effort is contractible\(^2\)
- Because (PC) is binding, the agent’s expected utility equals his reservation utility, which is as in case effort is contractible
- Since the principal has to compensate the agent for bearing risks (i.e., pay him a risk premium), the principal’s expected utility is lower than in case effort is contractible or in case effort is not contractible and the agent is risk neutral

\(^2\)Note that the first-best effort depends on the agent’s risk preferences, i.e., it is different if the agent is risk averse than if he is risk neutral
3.4.1 Additional information
Additional information

- Suppose that the principal can make the agent’s remuneration not only dependent on the outcome/existing performance measure $m$, but also on the additional performance measure $k$, which can take values $\{1, \ldots, K\}$
  - For example, $k$ (or $r_k$) may be the output of another agent, a quality measure, a measure of customers’ satisfaction, or the growth of sales

- Formally, the probability that $(m, k)$ realizes if the agent invests effort $e$ is $p_{m,k}(e)$, in which case the wage payment is $w_{m,k}$

- The previous analysis stays valid also with two performance measures

- We thus get the following modified Holmström formula:

\[
\frac{1}{u'(w_{m,k})} = \lambda + \mu \frac{p'_{m,k}(e)}{p_{m,k}(e)} \quad (\text{MHF})
\]
When is it optimal to make the agent’s compensation a function not only of $m$, but also of $k$?

The answer is straightforward: When the likelihood ratio
\[
\frac{p'_{m,k}(e)}{p_{m,k}(e)}
\]
depends on $k$

Conversely, when the likelihood ratio is independent of $k$, then there is no gain from contracting on $k$

- Indeed, it would be sub-optimal in this case because such a compensation scheme would fail to satisfy (MHF)
- Intuition: Contracting on $k$ only adds risk, which increases the risk premium demanded by the agent
**Proposition 4**

Suppose there are two measures for the agent’s performance. The agent’s compensation, i.e., the optimal contract, depends not only on the outcome \( m \), but also on \( k \), if and only if the likelihood ratio

\[
\frac{p'_{m,k}(e)}{p_{m,k}(e)}
\]

depends on \( k \).

**Proposition 5**

The likelihood ratio \( \frac{p'_{m,k}(e)}{p_{m,k}(e)} \) is independent of the additional performance measure \( k \) if and only if there exists a function \( h(m, k) \) such that

\[
p_{m,k}(e) = h(m, k)p_m(e)
\]

holds for all \((m, k)\) and all \( e \).
Proof of Proposition 5

- We first show that if there exists a function $h(m, k)$ such that $p_{m,k}(e) = h(m, k)p_m(e)$ holds for all $(m, k)$ and all $e$, then $\frac{p'_{m,k}(e)}{p_{m,k}(e)}$ is independent of $k$.

- Suppose there exists a function $h(m, k)$ such that $p_{m,k}(e) = h(m, k)p_m(e)$ holds for all $(m, k)$ and all $e$.

- Then
  \[
  \frac{p'_{m,k}(e)}{p_{m,k}(e)} = \frac{h(m, k)p'_m(e)}{h(m, k)p_m(e)} = \frac{p'_m(e)}{p_m(e)}
  \]

- The likelihood ratio $\frac{p'_{m,k}(e)}{p_{m,k}(e)}$ is thus independent of the realizations of $k$. 


We next show that if the likelihood ratio $\frac{p_{m,k}^{'}(e)}{p_{m,k}(e)}$ is independent of the realizations of the additional performance measure $k$, i.e.,

$$\frac{p_{m,k}^{'}(e)}{p_{m,k}(e)} = \frac{p_{m}^{'}(e)}{p_{m}(e)}$$

holds for all $(m, k)$ and all $e$, then a function $h(m, k)$ exists such that $p_{m,k}(e) = h(m, k)p_{m}(e)$ holds for all $(m, k)$ and all $e$.

The integral of $\frac{p_{m,k}^{'}(e)}{p_{m,k}(e)}$ is $\ln p_{m,k}(e) + constant_1$ and the integral of $\frac{p_{m}^{'}(e)}{p_{m}(e)}$ is $\ln p_{m}(e) + constant_2$ and thus

$$\frac{p_{m,k}^{'}(e)}{p_{m,k}(e)} = \frac{p_{m}^{'}(e)}{p_{m}(e)}$$

$$\Rightarrow \ln p_{m,k}(e) + constant_1 = \ln p_{m}(e) + constant_2$$

$$\Rightarrow \ln \left( \frac{p_{m,k}(e)}{p_{m}(e)} \right) = constant_3$$

$$\Rightarrow \frac{p_{m,k}(e)}{p_{m}(e)} = \exp\{constant_3\} = h(m, k) \quad \square$$
Interpretation

- If this condition holds, then we say that $m$ is a sufficient statistic for effort $e$ given $(m, k)$

- Thus, if the condition holds and we know $m$, knowing additionally $k$ is uninformative about effort $e$

- The optimal contract should thus be based on all performance measures that convey information about the agent’s decision (this lowers the risk premium)

- But it is not desirable to include performance measures that are statistically redundant with other measures (this would increase the risk premium)
Additional information and limited liability

- Note that these arguments (with slight changes) also apply to the model with risk neutrality and limited liability
  - Using all informative performance measures minimizes the agent’s rent
- However, with risk neutrality and unlimited liability or with contractible effort the first-best effort is implemented to first-best wage costs even without additional performance measures, which is why there is no reason to use an additional measure
Relative performance evaluation

- There are often several agents that do similar jobs
- These agents’ outcomes/performances are then affected by similar exogenous shocks like the business cycle or weather
  - E.g., two agents sell ice cream
- The optimal contract for each agent should include measures of the exogenous shocks or the other agents’ outcomes/performances
- This is done in practice:
  - Top managers of one firm are evaluated in comparison to other top managers of other firms within the same industry
  - Employees that work in the same firm are evaluated in comparison to each other
Problem in some contexts: relative performance evaluation creates competition which could backfire

- Agents have then little incentives to help each other
- On the contrary, they may try to sabotage each other
- Agents may collude, i.e., all invest very little effort
- Agents may try to get colleagues with low abilities
3.4.2 LEN Model
LEN model

- A particularly simple model, which is convenient to analyze, is the LEN model.
- LEN model: linear contracts, exponential utility, and a normally distributed noise term.
- The return is \( r = e + \varepsilon \), where \( \varepsilon \sim N(0, \sigma_\varepsilon^2) \).
- The wage is \( w = t + sr \), \( (s, t) \in \mathbb{R}^2 \).
  - \( t \): fixed transfer or base wage
  - \( s \): variable share of return
- Agent’s utility is
  \[
  u_A = -\exp\{-\eta[w - c(e)]\},
  \]
  where \( \eta > 0 \) is the coefficient of absolute risk aversion (higher parameter corresponds to higher degree of risk aversion).
- For simplicity, we suppose that \( c(e) = \frac{1}{2} \psi e^2 \), with \( \psi > 0 \).
Agent’s expected utility

- We first want to determine the agent’s expected utility
- Let \( x := w - c(e) \)
- Then \( u_A = u(x) = -\exp\{-\eta x\} \)
- Note that \( \lim_{w \to \infty} u_A = 0 \) such that we need to have \( u_A^{\text{res}} < 0 \)
- \( \varepsilon \) is normally distributed \( \Rightarrow r \) is n.d. \( \Rightarrow w \) is n.d. \( \Rightarrow x \) is n.d.
- Since \( x \) is normally distributed (say with mean \( \mu \) and variance \( \sigma^2 \)) the density function is
  \[
  f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\}
  \]
The expected utility is thus

\[ E[u_A] = E[u(x)] = \int_{-\infty}^{\infty} -\exp\{-\eta x\} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\} \, dx \]

We can rewrite the right-hand side as

\[ -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\eta x - \frac{(x - \mu)^2}{2\sigma^2} \right\} \, dx \]

\[ = -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2} \left( 2\sigma^2\eta x + x^2 - 2x\mu + \mu^2 \right) \right\} \, dx \]

Defining \( \hat{\mu} := \mu - \sigma^2\eta \), we can rewrite

\[ -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2} \left( -2x\hat{\mu} + x^2 + \mu^2 - \hat{\mu}^2 + \hat{\mu}^2 \right) \right\} \, dx \]

\[ = -\exp\left\{ -\frac{1}{2\sigma^2} (\mu^2 - \hat{\mu}^2) \right\} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(x - \hat{\mu})^2}{2\sigma^2} \right\} \, dx \]
Under the integral we have a density function of a random variable with mean $\hat{\mu}$ and variance $\sigma^2$

The integral is thus 1

We can then rewrite

$$ E[u_A] = E[u(x)] = - \exp \left\{ - \frac{1}{2\sigma^2} (\mu^2 - \hat{\mu}^2) \right\} \cdot 1 $$

$$ = - \exp \left\{ -\mu \eta + \frac{\eta^2 \sigma^2}{2} \right\} $$
Certainty equivalent

- The agent’s certainty equivalent $CE$ (i.e., the fixed amount that yields the agent the same expected utility as the lottery induced by the contract) is

$$-\exp\{-\eta CE\} = -\exp \left\{-\mu \eta + \frac{\eta^2 \sigma^2}{2} \right\}$$

- We can solve for $CE = \mu - \frac{\eta \sigma^2}{2}$

- From above we get that

$$x = w - c(e) = t + sr - \frac{1}{2} \psi e^2 = t + s(e + \varepsilon) - \frac{1}{2} \psi e^2$$

- Remember that $\mu = E[x]$ and $\sigma^2 = \text{Var}[x]$

- We thus have that $\mu = t + se - \frac{1}{2} \psi e^2$ and $\sigma^2 = s^2 \sigma^2_{\varepsilon}$

- Therefore, $CE = t + se - \frac{1}{2} \psi e^2 - \frac{\eta s^2 \sigma^2_{\varepsilon}}{2}$
Agent’s problem

• The agent maximizes his expected utility $E[u_A]$ over his effort choice

• Maximizing $E[u_A]$ over effort $e$ is equivalent to maximizing $u(CE)$ over effort $e$, which (due to $u' > 0$) yields the same solution $e^*$ as maximizing $CE$ over effort $e$

• This is intuitive: Since the certainty equivalent is a monotonic transformation of expected utility, both represent the same preferences

• An equivalent problem of the agent is thus to maximize his certainty equivalent $CE$ over his effort choice

• This directly yields that

$$e = \begin{cases} s/\psi & \text{for } s \geq 0, \\ 0 & \text{otherwise} \end{cases}$$
Principal’s problem

- The principal
  \[
  \max_{(s,t)} E[u_P] \text{ subject to } (PC), (IC)
  \]
- The principal’s expected utility is
  \[
  E[u_P] = E[r - w] = E[(1 - s)(e + \varepsilon) - t] = (1 - s)e - t
  \]
- We suppose that \( s \geq 0 \)
- We solve the relaxed problem, where the agent’s effort choice is
  \( e = s/\psi \)
- The agent’s participation constraint is \( CE \geq w^{res} \), where \( w^{res} \) is given by
  \[
  -\exp\{-\eta w^{res}\} = u^{res}_A
  \]
- \( w^{res} \) is the monetary value of the agent’s outside option
Lagrangian

\[ \mathcal{L}(e, s, t) = (1 - s)e - t \]
\[ + \lambda \left( t + se - \frac{1}{2} \psi e^2 - \frac{\eta s^2 \sigma^2}{2} - w^{res} \right) \]
\[ + \mu \left( \frac{s}{\psi} - e \right) \]
Exercise 3.1

Solve the Lagrangian and determine the optimal contract \((s^*, t^*)\) as well as the optimally implemented effort level \(e^*\)
The optimal contract is $s^* = \frac{1}{1 + \psi \eta \sigma^2}$ and $t^* = \frac{\eta \sigma^2 + 1/\psi}{2(1 + \psi \eta \sigma^2)^2} + w^{res}$

And the optimally implemented effort level is $e^* = \frac{1}{\psi} \cdot \frac{1}{1 + \psi \eta \sigma^2}$

Note that $s^* > 0$, so that it was indeed justified to solve the relaxed problem.

Moreover, $s^* < 1$

This reflects the principal’s trade-off between providing insurance to the agent (maximal insurance is yielded for $s = 0$) and ...

... achieving a high level of efficiency (the principal could implement the efficient effort level $e^{effi} = 1/\psi$ by setting $s = 1$)
Exercise 3.2
Determine and interpret the comparative statics of $s^*$ with respect to $\eta$ and $\sigma^2_\varepsilon$. 
Discussion

- A major drawback of the LEN model is that it exogenously assumes that contracts are linear, but does not endogenously derive that linear contracts are optimal.

- As can be seen from (HF), although the optimal contract could be linear, there is no reason that it is linear in general.
  - Idea: Let $M = 3$ and take $r_1$ and $r_2$ as given; such that a linear contract is optimal, $r_3$ must take a specific value.

- Due to its simplicity (once one has derived the certainty equivalent), the LEN model is nonetheless often used in the literature.

- There are extensions of the basic model that guarantee that the optimal contract is linear.
  - E.g., Holmström and Milgrom (1987)
3.5 Multiple Tasks
Motivation

- Many agents handle multiple tasks, not only one task
- Examples:
  - A teacher cares that children learn math, but also cares for their social abilities
  - A manager cares for the current business, but also develops future markets
  - A worker tries to produce a large output, but is also concerned about quality
- If the tasks are independent, then we could treat them separably
- But tasks could be related via the production function or the agent’s effort cost function
Simple model

- There is a risk-neutral agent and two tasks, 1 and 2
- $e_1, e_2$: efforts invested in tasks 1 and 2, with $e_1, e_2 \geq 0$
- Effort measures the probability that a task is completed successfully
- For both tasks, the return is 1 in case of success and 0 in case of failure
- Only outcome of task 1 is contractible
- A contract thus takes the form $(w_1, w_2)$, where $w_1 [w_2]$ is the wage in case task 1 is completed with failure $m = 1$ [success $m = 2$]
- Limited liability: $w_1, w_2 \geq 0$
- Agent’s reservation utility is $u_{A}^{res} = 0$
Agent’s utility

- The agent is intrinsically motivated for task 2
  - He receives an additional utility of $B \geq 0$ in case task 2 is completed with success ...
  - ... and 0 in case of failure

- Agent’s effort costs are $c(e_1, e_2) = \frac{1}{2} \psi_1 e_1^2 + \frac{1}{2} \psi_2 e_2^2 + ge_1 e_2$

- We concentrate on the case where tasks are substitutes, $g > 0$, such that investing more effort in one task increases the marginal effort costs of the other task

- So that effort costs are convex, we assume that $\psi_1 \psi_2 - g^2 > 0$

- It is never optimal for the agent to invest an effort level above one (investing one unit of effort already guarantees success)

- Agent’s expected utility is thus
  
  $$E[u_A] = e_1 w_2 + (1 - e_1) w_1 + e_2 B - c(e)$$
Principal’s utility

- Principal’s expected utility is
  \[ E[u_P] = e_1(1 - w_2) + (1 - e_1)(0 - w_1) + e_2 I + (1 - e_2)0 \]

- Like in the basic limited liability model, the principal optimally sets
  \[ w_1^* = 0 \]
Agent’s effort choice

- Ignoring that $e_i \in [0,1]$, the agent’s first-order conditions are
  \[
  \frac{\partial E[u_A]}{\partial e_1} = w_2 - \psi_1 e_1 - g e_2 = 0
  \]
  \[
  \frac{\partial E[u_A]}{\partial e_2} = B - \psi_2 e_2 - g e_1 = 0
  \]

- The agent’s effort choices are thus
  \[
  e_1 = \min \left\{ \max \left\{ 0, \frac{\psi_2 w_2 - gB}{\psi_1 \psi_2 - g^2} \right\} , 1 \right\}
  \]
  \[
  e_2 = \min \left\{ \max \left\{ 0, \frac{\psi_1 B - gw_2}{\psi_1 \psi_2 - g^2} \right\} , 1 \right\}
  \]
Results and discussion

- A higher reward to complete task 1, namely a higher $w_2$, (weakly) increases the effort the agent invests in task 1, $e_1$
- However, by (weakly) raising the marginal effort costs for task 2, a higher $w_2$ (weakly) decreases the effort the agent invests in task 2, $e_2$
- Note that if efforts are interior, so $e_1, e_2 \in (0,1)$, then the above statements are strict
- Therefore, higher incentives need not raise the total expected return $e_1 + e_2$
- Incentives could thus be counterproductive
When determining the incentives for task 1, via the choice of $w_2$, the principal must not only take into account the beneficial effect on task 1, but also the harmful effect on task 2.

The harmful effect can be so large that the principal optimally sets no incentives at all.

Example: $\psi_1 = \psi_2 = 1$ and $g = B = 1/2$.

As the graphic on the next slide reveals, the principal optimally sets no incentives at all, i.e., chooses $w^*_2 = 0$. 
Figure: The principal’s expected utility $E[u_P]$ is on the vertical axis and the incentives $w_2$ are on the horizontal axis.
Practical examples

- Rewarding teachers for good grades of their students incentivizes teachers to focus on narrowly defined basic skills that are tested, i.e., “teach to the test”, at the expense of a broader curriculum or the development of students’ social skills.

- If quality is important and could not or only poorly be measured, a firm cannot maintain good quality if it rewards quantity, for example, via using piece-rates.
Remarks

- Note that one could also consider a model with risk aversion and unlimited liability to discuss the issue of multiple tasks.
- This is what the seminal paper of the literature on multiple tasks, Holmström and Milgrom (1991), does.
3.6 Behavioral Economics and Incentives
Motivation

- We have seen that incentives can be counterproductive when the agent has to care for multiple tasks.
- We now show that also behavioral economics offers interesting explanations why incentives can fail to produce better outcomes.
- For an overview, see Kamenica (2012).
Contextual inference

- Suppose that in the absence of any monetary incentives you would solve two crossword puzzles a week.
- If offered $5 per puzzle solved, however, you would solve no puzzles.
- Some studies demonstrate precisely this type of effect.
- How can we explain this behavior?
- Contextual inference: People are often unsure about what the best action is and consequently seek clues from the environment.
- If you are unsure whether solving puzzles is fun and someone offers you $5 per puzzle solved, you might reasonably infer that this activity is not enjoyable and thus forgo it.
Signaling

- Offering a payment for a noble task can backfire
- For example, paying for blood donation may reduce the number of people willing to give blood
- Basic idea of signaling-models:
  - Many people care about other people’s beliefs about how altruistic they are
  - Without monetary incentives, engaging in the task is a signal that the person is altruistic
  - Introducing monetary incentive reduces the credibility of the signal
  - Consequently, monetary incentives can reduce the willingness to do good
Also fees can backfire

Gneezy and Rustichini (2000) introduced a fine for parents coming late to pick up their children in Israeli day-care centers

The number of parents who showed up late increased
3.7 Fixed wages
Fixed wages and effort

- In reality, not all employees have performance-dependent wages, i.e., their wages are fixed.
- Without monetary incentives, these employees/agents should choose to invest zero effort (no matter how large the fixed wage is).
- We next explain that this is not necessarily true in several scenarios.
Scenario 1: Efficiency wage theory

- Employees often receive substantial fixed wages, but are fired if their performance is rather bad.
- But then only the short-run wages are fixed, not the long-run wages.
- Employees invest effort to lower the probability of being fired.
- But the threat of being fired is only effective if an employee’s wage is rather high.
- Thus, an employer may choose to offer a rather high wage (exceeds the equilibrium wage on the labor market).
- Since employers may choose wages above the market-clearing level, the efficiency wage theory can also explain involuntary unemployment.
Empirical evidence

- Henry Ford introduced the five dollar day in the year 1914
- Most worker’s wages doubled
- Ford’s profit nonetheless increased
- Reason: Substantial increase in productivity due to better worker morale (i.e., more effort) and lower turnover
Scenario 2: Career concerns

- We next sketch a simple model which is due to Holmström (1982)
- Agents live and work for several periods
- Principals cannot write contracts that depend on agents’ outcomes
- Possible reasons: (i) outcomes are not verifiable or (ii) agents can only be paid in advance
- Agents’ outcomes are, however, observable
- Agents’ outcomes depend on their talents and efforts
- Only an agent knows his talent and his effort investment
- Each agent invests effort in all but the last period to increase his expected outcome because
  - ...a better outcome increases the principals’ perceptions about the agent’s talent
  - ...and thus the agent’s future wages
The agent thus invests effort to positively influence his reputation for being a talented agent, i.e., to positively influence his career.

Note that this disciplining device is only an imperfect substitute for explicit incentives contingent on outcome/performance.

In practice, the implicit incentives created by career concerns could nonetheless be quite powerful: young scientists (doctoral students, Postdocs, Juniorprofessors) usually work quite hard although there are few explicit incentives.

General lesson for contract design: when designing contracts with explicit incentives, one should take into account the agent’s implicit incentives created by career concerns.
Scenario 3: Intrinsic motivation

- In reality, many people are intrinsically motivated.
- Such agents invest at least some effort if they are not incentivized monetarily.
- One should thus interpret effort $e$ as the incentivized level of effort which an agent spends in excess of the level which he chooses due to his intrinsic motivation.
3.8 General remarks
Remark 1: Bargaining power

- We have assumed that the principal has all the bargaining power (she is able to design the contract, while the agent is only able to accept or reject it)

- This is unrealistic in many real-world situations; e.g., Lufthansa vs its pilots

- How do the results change if the agent has some bargaining power?
Due the technology and the contracting environment, only a certain set of expected utilities, i.e., combinations of $E[u_P]$ and $E[u_A]$, is attainable.

- We know from before that with (i) contractible effort or with (ii) non-contractible effort, risk neutrality, and full liability of the agent the contracting environment imposes no additional restriction on the set of attainable expected utilities; then the set of attainable expected utilities is given only by the technology.
- This is in stark contrast to the cases of (i) non-contractible effort and risk aversion or (ii) non-contractible effort and limited liability; then the contracting environment shrinks the set of attainable expected utilities.
We next consider alternative cases and suppose that the agent’s liability is unlimited.

Case 1: the principal has all bargaining power

- Her problem is then to maximize her expected utility subject to the constraint that the implemented combination of $E[u_P]$ and $E[u_A]$ has to belong to the set of attainable expected utilities.
- Note that only the Pareto frontier of the set of attainable expected utilities is important.
- The principal yields the highest $E[u_P]$ when she chooses the point on the Pareto frontier where $E[u_A] = u_{A}^{res}$.
- The expected utilities which are implemented in case 1 are denoted by $E[u_A]_1$ and $E[u_P]_1$. 
Case 2: the principal has not all bargaining power

- We restrict our attention to bargaining processes where the outcome is on the Pareto frontier of the set of attainable expected utilities (the idea is that for a bargaining outcome which is not on the Pareto frontier, players can improve by renegotiate to an outcome on the Pareto frontier)

- Since the principal has no longer all of the bargaining power it is reasonable to have $E[u_A]_2 > E[u_A]_1$ and $E[u_P]_2 < E[u_P]_1$
Case 3: the principal has all the bargaining power, but we change the agent’s reservation utility to \( u_{A}^{\text{resnew}} = E[u_{A}]_{2} \)

- The principal will then choose the point on the Pareto frontier where \( E[u_{A}]_{3} = u_{A}^{\text{resnew}} \)

- We thus have that \( E[u_{A}]_{3} = E[u_{A}]_{2} \) and, since we are on the Pareto frontier in cases 2 and 3, \( E[u_{P}]_{3} = E[u_{P}]_{2} \)

Thus, a change of the bargaining power has the same effect as a change of the agent’s reservation utility
• The case where the principal has all bargaining power is especially convenient to analyze.

• Thus, one usually assumes that the bargaining power is in the hand of the principal and just increases $u_A^{\text{res}}$ to explore cases where the agent has bargaining power.

• If the agent is protected by limited liability, we have to modify the above arguments slightly, but the result stays the same.
Remark 2: Participation of the principal

- We have not considered the principal’s participation constraint.
- If the technology and the contracting environment is such that the set of attainable expected utilities does not allow the principal to yield an expected utility that is at least as high as her reservation utility, then the principal will not participate.
- Since the principal will just leave the relationship with the agent, she does not construct a contract.
- But this is then not interesting for us.
- We thus concentrate on the case where the principal is able to achieve an expected utility that is at least as high as her reservation utility.
Remark 3: Repeated moral hazard

- Suppose the principal and the agent interact for several periods, the agent is risk averse, and effort is not contractible.
- Should the principal just repeat the contract that is she optimally calculated for the case of a one-time interaction?
- No, the principal can do better by designing a contract with memory, where a high outcome in the current period does not only cause a high current wage, but also higher future wages.
- Idea:
  - Suppose there are two periods, two possible outcomes in every period, and the principal repeats the contract that is optimal for a one-time interaction.
  - If the principal increases the agent’s second-period wages after a successful first-period outcome, but lowers the agent’s second-period wages after an unsuccessful first-period outcome, this creates additional incentives for the agent in the first period, with negligible (second-order) costs.
- For details, see Chapter 8 of Laffont and Martimort (2001).
Exercises
Exercise 3.3

- Suppose there is asymmetric information, the agent is protected by limited liability, and that \( p(e) = e, \ c(e) = e^2/2, \) and \( \Delta r, e \in [0,1] \)
- What effort level does the principal optimally implement?
- What is the optimal contract?
- What is the first-best effort level?
Solution Exercise 3.3

- We know that the optimal values $\Delta w^*$ and $e^*$ solve ($IC'$), which is

$$p'(e)\Delta w = c'(e)$$

and

$$p'(e)(\Delta r - \Delta w) + \frac{p(e)}{p'(e)}(p''(e)\Delta w - c''(e)) = 0$$

- This directly yields that $\Delta w^* = e^* = \Delta r/2$

- The optimal contract is thus $(w_1^* = 0, w_2^* = \Delta r/2)$

- Maximizing $p(e)r_2 + (1 - p(e))r_1 - c(e)$ over $e$ yields

$$e^{effi} = e^{FB} = \Delta r$$
Exercise 3.4

• Suppose that, as in Exercise 3.3, \( p(e) = e, \ c(e) = e^2 / 2, \) and \( \Delta r, e \in [0, 1] \)

• Under the cost-minimizing contract \( w_1^* = 0 \)

• Determine the agent’s rent and the principal’s expected utility in dependence of \( w_2 \)

• What values do these take for \( w_2 = 0 \) and what for \( w_2 = \Delta r \)?

• Interpret the results and compare these to the case where the principal chooses the optimal contract
Solution Exercise 3.4

- From (IC’) we directly get that the agent chooses effort
  \[ e = \Delta w = w_2 \]

- The agent’s rent is
  \[ E[u_A] - u_A^{res} = w_2^2 - w_2^2/2 - 0 = \frac{w_2^2}{2} \]

- The principal’s expected utility is
  \[ E[u_P] = r_1 + w_2(\Delta r - w_2) \]

- For \( w_2 = 0 \) we get \( E[u_A] - u_A^{res} = 0 \) and \( E[u_P] = r_1 \)

- Interpretation: If \( w_2 = 0 \), the wage spread \( \Delta w \) is zero. So a high return \( r_2 \) is equally well remunerated than a low return \( r_1 \). Thus, the agent optimally invests zero effort. Given this, the principal receives the low return \( r_1 \) for sure (recall that here \( p(0) = 0 \)) and has no wage costs, which is why her expected utility is \( r_1 \). Since \( w_2 = 0 \), the agent receives no rent.
For $w_2 = \Delta r$ we have $E[u_A] - u_A^{res} = \frac{\Delta r^2}{2}$ and $E[u_P] = r_1$

Interpretation: If $w_2 = \Delta r$, the agent is the residual claimant. He thus chooses the first-best effort level. Since the incentives are positive, the agent receives a positive rent. Due to limited liability, the principal cannot charge the agent for making him the residual claimant. That is, $w_1 < 0$ is not possible. In case the outcome turns out to be high, she has to give the additional return $\Delta r$ to the agent. This is why her expected utility equals the low return $r_1$ even in case the outcome is high.
From Exercise 3.1 we know that the principal optimally chooses 
\[ w_2^* = \Delta r / 2 \]

The agent’s rent is then \( E[u_A] - u_A^{\text{res}} = \frac{\Delta r^2}{8} \)

The principal’s expected utility is then \( E[u_P] = r_1 + \frac{\Delta r^2}{2} \)

By setting the optimal contract for her, the principal clearly benefits, compared to the cases with \( w_2 = 0 \) or \( w_2 = \Delta r \)

Since the agent’s rent is increasing in \( w_2 \), the agent is better off with \( w_2 = \Delta r / 2 \) than with \( w_2 = 0 \), but worse off with \( w_2 = \Delta r / 2 \) than with \( w_2 = \Delta r \) \( \square \)
Exercise 3.5
Consider the following standard model of moral hazard, where agent’s effort is binary:

- The agent can either work (invest effort $e = 1$) or shirk (invest effort $e = 0$), i.e., effort is binary
- Investing effort causes costs for the agent of $c(e)$, where $c(1) > c(0) = 0$
- There are two possible outcomes levels, $r_1$ and $r_2$, where $r_1 < r_2$
- The probability that the return out to be $r_2$ is $p(e)$, that for $r_1$ is $1 - p(e)$, where $p(1) > p(0)$
- A contract then takes the form $(w_1, w_2)$, where $w_1$ is the wage in case outcome 1 is realized and $w_2$ the wage in case outcome 2 is realized
- The agent’s utility is
  \[ u_A = w - c(e) \]
- The principal’ utility is
  \[ u_P = q - w \]
Timing

1. The principal suggests a contract \((w_1, w_2)\)

2. The agent can either accept or reject
   - If the agent rejects, he receives the reservation utility \(u^{res}_A = 0\)
   - If the agent accepts, he chooses an effort level \(e \in \{0, 1\}\)

3. The outcome is determined and the payment is made according to the contract
Questions

a) Determine the welfare maximum

b) Solve the principal’s problem for the case where effort is contractible

c) Solve the principal’s problem for the case where effort is not contractible
Suppose there is a social planner who maximizes the expected welfare and can choose which effort the agent selects.

She thus maximizes

\[ E[V] := E[u_A] + E[u_P] = w - c(e) + E[q] - w \]

\[ = E[q] - c(e) = p(e)r_2 + (1 - p(e))r_1 - c(e) \]

- If the agent works, \( E[V]|_{e=1} = p(1)r_2 + (1 - p(1))r_1 - c(1) \)
- If the agent shirks, \( E[V]|_{e=0} = p(0)r_2 + (1 - p(0))r_1 \)

Comparison: It is efficient that the agent works if

\[ E[V]|_{e=1} \geq E[V]|_{e=0} \iff \Delta p \Delta r \geq c(1) \]

where \( \Delta r := r_2 - r_1 \) and \( \Delta p := p(1) - p(0) \)

Thus, \( e^{effi} = 1 \) if \( \Delta p \Delta r \geq c(1) \) and \( e^{effi} = 0 \) otherwise
b) contractible effort

- With symmetric information, the principal can specify by a contract which effort the agent must invest.
- A contract hence specifies a wage payment $w$ and an effort $e$.
- If the principal wants that the agent works, her problem is to maximize

$$E[u_P] = p(1)r_2 + (1 - p(1))r_1 - w$$

subject to the agent’s participation constraint

$$u_A \geq 0 \iff w - c(1) \geq 0$$  

(PC)

- The Lagrangian writes

$$\mathcal{L}(e = 1, w) = p(1)r_2 + (1 - p(1))r_1 - w + \lambda(w - c(1))$$

- $\frac{\partial \mathcal{L}(e=1,w)}{\partial w} = -1 + \lambda = 0$, so $\lambda^{FB} = 1$ and the (PC) must bind.
- Thus, $w^{FB} = c(1)$ and $E[u_P]|_{e=1} = p(1)r_2 + (1 - p(1))r_1 - c(1)$.
If the principal wants that the agent shirks, her problem is to maximize

\[ E[u_P] = p(0)r_2 + (1 - p(0))r_1 - w \]

subject to the agent’s participation constraint

\[ u_A \geq 0 \iff w \geq 0 \quad \text{(PC)} \]

The Lagrangian writes

\[ L(e=0, w) = p(0)r_2 + (1 - p(0))r_1 - w + \lambda(w) \]

\[ \frac{\partial L(e=0, w)}{\partial w} = -1 + \lambda = 0, \quad \text{so} \quad \lambda^{FB} = 1 \quad \text{and the (PC) must bind} \]

Thus, \( w^{FB} = 0 \) and \( E[u_P]|_{e=0} = p(0)r_2 + (1 - p(0))r_1 \)

Comparison: The principal optimally implements \( e = 1 \) if

\[ E[u_P]|_{e=1} \geq E[u_P]|_{e=0} \iff \Delta p \Delta r \geq c(1) \]

This is the same condition as in benchmark I, where a social planner maximizes the expected welfare, thus \( e^{FB} = e^{effi} \).
c) non-contractible effort

- The contract now takes the form \((w_1, w_2)\)
- We suppose that the agent’s liability is unlimited, i.e., the wages could be arbitrarily low
- If the principal wants that the agent works, her problem is to maximize

\[
E[u_P] = p(1)(r_2 - w_2) + (1 - p(1))(r_1 - w_1)
\]

subject to the agent’s participation constraint

\[
E[u_A] \geq 0 \iff p(1)w_2 + (1 - p(1))w_1 - c(1) \geq 0 \quad \text{(PC)}
\]

and the incentive constraint

\[
p(1)w_2 + (1 - p(1))w_1 - c(1) \geq p(0)w_2 + (1 - p(0))w_1 \quad \text{(IC)}
\]
The Lagrangian writes

\[ \mathcal{L}(e = 1, w_1, w_2) = p(1)(r_2 - w_2) + (1 - p(1))(r_1 - w_1) \]
\[ + \lambda(p(1)w_2 + (1 - p(1))w_1 - c(1)) \]
\[ + \mu(p(1)w_2 + (1 - p(1))w_1 - c(1) - p(0)w_2 - (1 - p(0))w_1) \]

Note that the incentive constraint is an inequality condition!
Optimum

In the optimum, the following holds:

\[
\frac{\partial L}{\partial w_1} (e = 1, w_1, w_2) = -(1 - p(1)) + \lambda(1 - p(1)) + \mu(-p(1) + p(0)) = 0
\]  
(19)

\[
\frac{\partial L}{\partial w_2} (e = 1, w_1, w_2) = -p(1) + \lambda p(1) + \mu(p(1) - p(0)) = 0
\]  
(20)

\[p(1)w_2 + (1 - p(1))w_1 - c(1) \geq 0\]  
(PC)

\[p(1)w_2 + (1 - p(1))w_1 - c(1) \geq p(0)w_2 + (1 - p(0))w_1\]  
(IC)

\[\lambda \geq 0; \text{ if (PC) does not bind, then } \lambda = 0\]  
(21)

\[\mu \geq 0; \text{ if (IC) does not bind, then } \mu = 0\]  
(22)
Solution

- Adding (19) and (20) yields $\lambda^* = 1$ and so (PC) binds.
- Plugging $\lambda^* = 1$ in (19) or (20) yields that $\mu^* = 0$.
- An optimal contract $(w_1^*, w_2^*)$ solves (PC) with $=\_=$ and satisfies (IC).
- Note that there are multiple optimal contracts.
- One optimal contract is the one where not only (PC), but also (IC) binds.
- Plugging the binding (PC) in the principal’s objective function yields that for an optimal contract

$$E[u_P]|_{e=1} = p(1)r_2 + (1 - p(1))r_1 - c(1)$$
If the principal wants that the agent shirks, her problem is to maximize

\[ E[u_P] = p(0)(r_2 - w_2) + (1 - p(0))(r_1 - w_1) \]

subject to the agent’s participation constraint

\[ E[u_A] \geq 0 \iff p(0)w_2 + (1 - p(0))w_1 \geq 0 \quad \text{(PC)} \]

and the incentive constraint

\[ p(0)w_2 + (1 - p(0))w_1 \geq p(1)w_2 + (1 - p(1))w_1 - c(1) \quad \text{(IC)} \]

The Lagrangian writes

\[
\mathcal{L}(e = 0, w_1, w_2) = p(0)(r_2 - w_2) + (1 - p(0))(r_1 - w_1) \\
+ \lambda(p(0)w_2 + (1 - p(0))w_1) \\
+ \mu(p(0)w_2 + (1 - p(0))w_1 - p(1)w_2 - (1 - p(1))w_1 + c(1))
\]
Optimization yields (check this by solving the Lagrange problem) that the optimal contract to implement \( e = 0 \) is \((w_1^* = 0, w_2^* = 0)\) such that
\[
E[u_P]|_{e=0} = p(0)r_2 + (1 - p(0))r_1
\]

Comparison: The principal optimally implements \( e^* = 1 \) if
\[
E[u_P]|_{e=1} \geq E[u_P]|_{e=0} \iff \Delta p \Delta r \geq c(1)
\]

This is the same condition as in benchmarks I and II, thus
\( e^* = e^{FB} = e^{effi} \)

This is the same result as with continuous effort

As with continuous effort, asymmetric information influences the results once the agent’s liability is limited or the agent is risk averse
Exercise 3.6

Consider a cashless entrepreneur who wants to borrow and carry out the following project. With an investment normalized to 1 unit he will get an return/output of $z$ with probability $\bar{P} > 0$ if he exerts an effort level of $\bar{e}$ and with probability $P > 0$ ($\bar{P} > P$) if he exerts no effort, and nothing otherwise. Let the cost of effort $\bar{e}$ for the entrepreneur be $\psi > 0$. Furthermore his status quo utility level is normalized to 0 and $Pz < r$.

A monopolistic bank with cost of fund $r$ offers a loan of 1 unit for a reimbursement of $x$ when the project is successful.

The entrepreneur is risk neutral and protected by limited liability. Determine the optimal loan contract of the bank in case effort is contractible and in case effort is not contractible.
Exercise 3.7

- A firm’s return can take two different levels: high, $r_2 = 6$, or low, $r_1 = 0$
- The firm’s profit is $\pi = r - w$, where $w$ is the wage payment to its worker
- Before contracting with its worker, the firm can choose between two production technologies
  1. Technology $X$: The probability that the outcome is high is $p^X(e) = 0.4 + 0.4e$
  2. Technology $Y$: The probability that the outcome is high is $p^Y(e) = 0.7e$
- The worker is risk neutral, protected by limited liability, has a reservation utility of zero, and has effort costs of $e$, where $e \in \{0,1\}$
Questions:

- Determine for each technology the firm’s optimal contract \((w_1^*, w_2^*)\) and its expected profit
- Which technology does the firm optimally choose?
- Interpret your results
Solution Exercise 3.7

- Consider first technology \( X \)
- Trivially, if the firm seeks to implement \( e = 0 \) then the cost-minimizing contract is \( w_1^X = w_2^X = 0 \)
- The firm’s expected wage payment is 0
- The firm’s expected return is \( 0,4 \cdot 6 = 2,4 \)
- The firm’s expected profit is

\[
E[\pi]_{e=0}^X = p^X(0)r_2 = 2,4
\]
If the firm wants to implement $e = 1$, then the worker’s incentive constraint has to be satisfied.

This can be written as $\Delta p^X \Delta w^X \geq c(1) = 1$, where $\Delta p^X := p^X(1) - p^X(0)$ and $\Delta w^X := w_2^X - w_1^X$.

We ignore the limited liability constraint $w_2^X \geq 0$ and thus only solve the relaxed problem.
The Lagrangian is

\[ \mathcal{L} \left( e = 1, w_1^x, w_2^x \right) = p^x(1)(r_2 - w_2^x) + (1 - p^x(1))(r_1 - w_1^x) \\
+ \rho(w_1^x - 0) \\
+ \mu(\Delta p^x \Delta w^x - 1) \]

which we can rewrite

\[ \mathcal{L} \left( e = 1, w_1^x, w_2^x \right) = 0,8(6 - w_2^x) + 0,2(0 - w_1^x) \\
+ \rho(w_1^x - 0) \\
+ \mu(0,4\Delta w^x - 1) \]

Differentiating yields

\[ \frac{\partial \mathcal{L} \left( e = 1, w_1^x, w_2^x \right)}{\partial w_1^x} = -0,2 + \rho - \mu0,4 = 0 \]

\[ \frac{\partial \mathcal{L} \left( e = 1, w_1^x, w_2^x \right)}{\partial w_2^x} = -0,8 + \mu0,4 = 0 \]
• Thus, $\mu^* = 2$ and $\rho^* = 1$

• $\rho^* > 0$ implies that the limited liability constraint $\omega_1^X \geq 0$ binds and thus that $\omega_1^X = 0$

• Recall that effort is a discrete variable and that the incentive constraint is an inequality constraint; $\mu^* = 2$ thus implies that the incentive constraints $\Delta \rho^X \Delta \omega^X \geq c(1) \iff 0,4 \Delta \omega^X \geq 1$ holds with equality and thus that $\omega_2^X = 2,5$

• The firm’s expected wage payment is $0,8 \cdot 2,5 = 2$

• The firm’s expected return is $0,8 \cdot 6 = 4,8$

• The firm’s expected profit is $E[\pi]_{e=1}^X = 2,8$
Consider now technology $Y$.

Similar calculations as for technology $X$ show that to implement effort $e = 0$ the cost-minimizing contract is $w_1^Y = w_2^Y = 0$

- The firm’s expected wage payment is 0.
- The firm’s expected return is 0.
- The firm’s expected profit is $E[\pi]_{e=0}^Y = 0$. 
The cost-minimizing contract to implement \( e = 1 \) is \( w_1^Y = 0 \) and \( w_2^Y = 1/0,7 \)

The firm’s expected wage payment is \( 0,7 \cdot 1/0,7 = 1 \)

The firm’s expected return is \( 0,7 \cdot 6 = 4,2 \)

The firm’s expected profit is \( E[\pi]^Y_{e=1} = 3,2 \)
• The firm thus optimally chooses technology $Y$ and implements $e^* = 1$ by the contract $(w_1^* = 0, w_2^* = 1/0.7)$

• It is remarkable that the firm chooses technology $Y$ instead of $X$, since for all efforts $e \in \{0,1\}$ the expected return is higher with $X$ than with $Y$

• The firm nonetheless chooses technology $Y$ since implementing $e = 1$ causes much lower expected wage costs with $Y$ than with $X$

• The expected wage costs to implement $e = 1$ are much lower with $Y$ than with $X$ because the likelihood ratio of the high outcome is

$$\frac{p^X(1) - p^X(0)}{p^X(1)} = 0.5$$

in case of technology $X$, but

$$\frac{p^Y(1) - p^Y(0)}{p^Y(1)} = 1$$

in case of technology $Y$
Exercise 3.8

- There are two agents, A and B, who are protected by limited liability.
- Agents’ reservation utility is zero.
- Each agent $i$ can choose between working, $e_i = 1$, and shirking, $e_i = 0$.
- Agent $i$’s effort costs are $e_i$.
- Each agent’s return is either high, $r^i_2$, or low, $r^i_1$.
- The probability that agent $i$’s return is high is
  \[ p(r^i_2|e_i, \kappa) = \frac{1}{2} + \frac{e_i}{10} + \kappa \]
- With equal probability, $\kappa$ is 0.4 or $-0.4$.
- $\kappa$ is the same for both agents.
- Players do not know $\kappa$ when concluding the contract and when choosing efforts.
• $\kappa$ is not contractible and can be interpreted as common noise (e.g., $\kappa$ may be the condition of a certain sector of the economy)

• Suppose the principal wants to implement that both agents work

• Agent $A$’s wage in case his outcome is $m$ and agent $B$’s is $k$ is denoted by $w_{m,k}$, with $m, k \in \{1,2\}$

• Taking as given that agent $B$ works, determine the cost-minimizing contract for $A$ and the resulting costs for the following four cases:
  1. Agent $A$’s remuneration can only depend on his own performance, so $w_{1,1} = w_{1,2}$ and $w_{2,1} = w_{2,2}$
  2. Agent $A$’s remuneration can depend on his own performance as well as on that of agent $B$
  3. There is a tournament, where agent $A$ earns $w_2$ when he produces a higher outcome than $B$, $w_2/2$ if the outcomes are the same, and zero otherwise
  4. Agent’s effort is contractible
Solution Exercise 3.8:

- We only sketch the solution
- We do not use Lagrangians to derive the optimum
- Instead, we exploit that the incentive constraint holds with equality in the optimum
- And that it is optimal only to reward the outcome with the highest likelihood ratio
Case 1

Throughout is optimal not to reward low performance, so $w_{1,k} = 0$ for $k \in \{1,2\}$

If $w_{2,1} = w_{2,2} =: \tilde{w}$, then agent $A$'s expected utility in case he works is

$$EU[u_A|e_A = 1] = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{10} - 0,4 \right) \tilde{w} + \frac{1}{2} \left( \frac{1}{2} + \frac{1}{10} + 0,4 \right) \tilde{w} - 1$$

In contrast, if $A$ shirks then his expected utility is

$$EU[u_A|e_A = 0] = \frac{1}{2} \left( \frac{1}{2} - 0,4 \right) \tilde{w} + \frac{1}{2} \left( \frac{1}{2} + 0,4 \right) \tilde{w}$$

Agent $A$ thus optimally works if and only if

$$EU[u_A|e_A = 1] \geq EU[u_A|e_A = 0] \iff \tilde{w} \geq 10 \quad (IC)$$

The cost minimizing principal thus chooses $\tilde{w}^* = 10$, resulting in expected wage costs for $A$ of $0,6 \cdot 10 = 6$
Case 2

Agent A’s expected utility in case he works is

\[
E[u_A|e_A = 1] = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{10} - 0.4 \right) \left( \frac{1}{2} + \frac{1}{10} - 0.4 \right) w_{2,2} \\
+ \frac{1}{2} \left( \frac{1}{2} + \frac{1}{10} - 0.4 \right) \left( 1 - \left( \frac{1}{2} + \frac{1}{10} - 0.4 \right) \right) w_{2,1} \\
+ \frac{1}{2} \left( \frac{1}{2} + \frac{1}{10} + 0.4 \right) \left( \frac{1}{2} + \frac{1}{10} + 0.4 \right) w_{2,2} \\
+ \frac{1}{2} \left( \frac{1}{2} + \frac{1}{10} + 0.4 \right) \left( 1 - \left( \frac{1}{2} + \frac{1}{10} + 0.4 \right) \right) w_{2,1} \\
- 1
\]

\[
= \frac{52}{100} w_{2,2} + \frac{8}{100} w_{2,1} - 1
\]
Agent A’s expected utility in case he shirks is

\[
E[u_A|e_A = 0] = \frac{1}{2} \left( \frac{1}{2} - 0.4 \right) \left( \frac{1}{2} + \frac{1}{10} - 0.4 \right) w_{2,2} \\
+ \frac{1}{2} \left( \frac{1}{2} - 0.4 \right) \left( 1 - \left( \frac{1}{2} + \frac{1}{10} - 0.4 \right) \right) w_{2,1} \\
+ \frac{1}{2} \left( \frac{1}{2} + 0.4 \right) \left( \frac{1}{2} + \frac{1}{10} + 0.4 \right) w_{2,2} \\
+ \frac{1}{2} \left( \frac{1}{2} + 0.4 \right) \left( 1 - \left( \frac{1}{2} + \frac{1}{10} + 0.4 \right) \right) w_{2,1} \\
= \frac{46}{100} w_{2,2} + \frac{4}{100} w_{2,1}
\]
Agent $A$ thus optimally works if and only if
\[
E[u_A|e_A = 1] \geq E[u_A|e_A = 0] \iff \frac{6}{100}w_{2,2} + \frac{4}{100}w_{2,1} - 1 \geq 0
\] (IC)

Since the principal has to pay the wage $w_{2,2}$ with probability $\frac{52}{100}$ while the wage $w_{2,1}$ with probability $\frac{8}{100}$, it is optimal for the principal to set $w_{2,2}^* = 0$

Technically, the likelihood ratio is
\[
\frac{E[p(m,k|e_A=1,\kappa)]-E[p(m,k|e_A=0,\kappa)]}{E[p(m,k|e_A=1,\kappa)]}
\]

- For the outcomes $(m = 2, k = 2)$ the likelihood ration is
  \[
  \frac{\frac{52}{100} - \frac{46}{100}}{\frac{52}{100}} = \frac{3}{26}
  \]
- For the outcomes $(m = 2, k = 1)$ the likelihood ration is
  \[
  \frac{\frac{8}{100} - \frac{4}{100}}{\frac{8}{100}} = \frac{1}{2}
  \]

The cost minimizing principal thus chooses $w_{2,1}^* = 100/4 = 25$, resulting in expected wage costs for $A$ of $\frac{8}{100} \cdot 25 = 2$
Case 3

Agent A’s expected utility in case he works is

\[ E[u_A|e_A = 1] = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{10} - 0.4 \right) \left( \frac{1}{2} + \frac{1}{10} - 0.4 \right) \frac{w_2}{2} 
\]

\[ + \frac{1}{2} \left( \frac{1}{2} + \frac{1}{10} - 0.4 \right) \left( 1 - \left( \frac{1}{2} + \frac{1}{10} - 0.4 \right) \right) w_2 
\]

\[ + \frac{1}{2} \left( \frac{1}{2} + \frac{1}{10} + 0.4 \right) \left( \frac{1}{2} + \frac{1}{10} + 0.4 \right) \frac{w_2}{2} 
\]

\[ + \frac{1}{2} \left( \frac{1}{2} + \frac{1}{10} + 0.4 \right) \left( 1 - \left( \frac{1}{2} + \frac{1}{10} + 0.4 \right) \right) w_2 
\]

\[ + \frac{1}{2} \left( 1 - \left( \frac{1}{2} + \frac{1}{10} - 0.4 \right) \right) \left( 1 - \left( \frac{1}{2} + \frac{1}{10} - 0.4 \right) \right) \frac{w_2}{2} 
\]

\[ + \frac{1}{2} \left( 1 - \left( \frac{1}{2} + \frac{1}{10} + 0.4 \right) \right) \left( 1 - \left( \frac{1}{2} + \frac{1}{10} + 0.4 \right) \right) \frac{w_2}{2} - 1 
\]

\[ = \frac{1}{2} w_2 - 1 
\]
Agent A’s expected utility in case he shirks is

\[
E[u_A|e_A = 0] = \frac{1}{2} \left( \frac{1}{2} - 0.4 \right) \left( \frac{1}{2} + \frac{1}{10} - 0.4 \right) \frac{w_2}{2} \\
+ \frac{1}{2} \left( \frac{1}{2} - 0.4 \right) \left( 1 - \left( \frac{1}{2} + \frac{1}{10} - 0.4 \right) \right) w_1 \\
+ \frac{1}{2} \left( \frac{1}{2} + 0.4 \right) \left( \frac{1}{2} + \frac{1}{10} + 0.4 \right) \frac{w_2}{2} \\
+ \frac{1}{2} \left( \frac{1}{2} + 0.4 \right) \left( 1 - \left( \frac{1}{2} + \frac{1}{10} + 0.4 \right) \right) w_2 \\
+ \frac{1}{2} \left( 1 - \left( \frac{1}{2} - 0.4 \right) \right) \left( 1 - \left( \frac{1}{2} + \frac{1}{10} - 0.4 \right) \right) \frac{w_2}{2} \\
+ \frac{1}{2} \left( 1 - \left( \frac{1}{2} + 0.4 \right) \right) \left( 1 - \left( \frac{1}{2} + \frac{1}{10} + 0.4 \right) \right) \frac{w_2}{2} \\
= \frac{45}{100} w_2
\]
Agent $A$ thus optimally works if and only if

$$E[u_A|e_A = 1] \geq E[u_A|e_A = 0] \iff w_2 \geq 20 \quad \text{(IC)}$$

The cost minimizing principal thus chooses $w_2^* = 20$, resulting in expected wage costs for $A$ of $5 \cdot 20 = 10$.
Case 4

- If effort is contractible, the principal optimally pays the agent his effort costs of 1
- The wage costs are thus 1
Exercise 3.9

A risk-averse agent initially possesses a wealth $W$. With probability $p(e)$ [respectively $1 - p(e)$] there will be an [respectively no] accident and his wealth is $W - s$ [respectively $W$]. The agent exercises care $e \in \{0, 1\}$ and $1 > p(0) > p(1) > 0$. Without insurance, when choosing care $e$, the agent’s expected utility is

$$u_A^{res}(e) = p(e)u(W - s) + (1 - p(e))u(W) - \psi e,$$

where the function $u$ satisfies $u(0) = 0$, $u' > 0$, $u'' < 0$, and $\psi e$ are the agent’s costs of care, with $\psi > 0$. Moreover, we assume that $u_A^{res}(1) > u_A^{res}(0)$.  


There is a risk-neutral insurance company who offers an insurance contract \((w_1, w_2)\), where \(w_1\) is the agent’s wealth in case of an accident and \(w_2\) in case of no accident. The company’s expected profit is 

\[
p(e)(W - s - w_1) + (1 - p(e))(W - w_2).
\]

The timing is as follows: The insurance company first suggests a contract, the agent accepts or rejects, then chooses care, and finally an accident happens or not and payments are made according to the contract (if there is one).

Determine the optimal contract to implement that the agent cares (i.e., chooses effort \(e = 1\)) for (i) the case of contractible care and (ii) for the case of non-contractible care.
Solution Exercise 3.9

- Without a contract with the insurance company the agent optimally takes care since \( u^\text{res}_A(1) > u^\text{res}_A(0) \)
- The agent’s reservation utility is thus \( u^\text{res}_A(1) \)
- When accepting the contract, the agent’s expected utility is
  \[
  E[u_A] = p(e)u(w_1) + (1 - p(e))u(w_2) - \psi e
  \]
- The agent’s participation constraint is thus
  \[
  E[u_A] \geq u^\text{res}_A(1) \quad \text{(PC)}
  \]
With contractible care/effort, the company’s problem is to maximize the expected profit subject to the agent’s participation constraint.

The Lagrangian is

\[ \mathcal{L}(e = 1, w_1, w_2) = p(1)(W - s - w_1) + (1 - p(1))(W - w_2) + \lambda (p(1)u(w_1) + (1 - p(1))u(w_2) - \psi - u^\text{res}_A(1)) \]

We get that

\[ \frac{\partial \mathcal{L}(e = 1, w_1, w_2)}{\partial w_1} = -p(1) + \lambda p(1)u'(w_1) = 0 \]

\[ \frac{\partial \mathcal{L}(e = 1, w_1, w_2)}{\partial w_2} = -(1 - p(1)) + \lambda (1 - p(1))u'(w_2) = 0 \]
We can rewrite these two equations as

$$\lambda u'(w_1) = 1$$
$$\lambda u'(w_2) = 1$$

Thus, $$w_1^* = w_2^*$$

Since $$\lambda^* > 0$$, the participation constraint binds

$$w_1^* = w_2^* = w^*$$ thus solves

$$u(w) = u_A^{res}(1) + \psi$$
• Consider next the case with non-contractible care/effort
• Recall that the company wants that the agent takes care, i.e., chooses effort $e = 1$
• The agent’s incentive constraint is thus

$$p(1)u(w_1) + (1 - p(1))u(w_2) - \psi \geq p(0)u(w_1) + (1 - p(0))u(w_2) \tag{IC}$$

• We can rewrite this as

$$\Delta p (u(w_1) - u(w_2)) - \psi \geq 0,$$

where $\Delta p := p(1) - p(0)$
• Note that $\Delta p < 0$
• The Lagrangian is

$$\mathcal{L} (e = 1, w_1, w_2) = p(1)(W - s - w_1) + (1 - p(1))(W - w_2)$$

$$+ \lambda (p(1)u(w_1) + (1 - p(1))u(w_2) - \psi - u_{A}^{res}(1))$$

$$+ \mu (\Delta p (u(w_1) - u(w_2)) - \psi)$$
In the optimum, the following holds:

\[
\frac{\partial \mathcal{L}(\cdot)}{\partial w_1} = -p(1) + \lambda p(1) u'(w_1) + \mu \Delta p u'(w_1) = 0 \tag{23}
\]

\[
\frac{\partial \mathcal{L}(\cdot)}{\partial w_2} = -(1 - p(1)) + \lambda (1 - p(1)) u'(w_2) - \mu \Delta p u'(w_2) = 0
\] (24)

\[p(1) u(w_1) + (1 - p(1)) u(w_2) - \psi - u_{\text{res}}(1) \geq 0 \tag{PC}\]

\[\Delta p (u(w_1) - u(w_2)) - \psi \geq 0 \tag{IC}\]

\[\lambda \geq 0; \text{ if (PC) does not bind, then } \lambda = 0 \tag{25}\]

\[\mu \geq 0; \text{ if (IC) does not bind, then } \mu = 0 \tag{26}\]
Since $0 < p(1) < 1$, $u' > 0$, $\mu \geq 0$, and $\Delta p < 0$, (23) implies that $\lambda^* > 0$

We next seek to determine $\mu^*$

The incentive constraint requires that $u(w_1) < u(w_2)$ and thus that $w_1 < w_2$

Intuition: We cannot reward the agent for having an accident if we want that he takes care

Together with $u'' < 0$ we thus have $u'(w_1) > u'(w_2)$

We multiply (23) by $(1 - p(1))$ and (24) by $p(1)$ to get

\[-p(1)(1-p(1))+\lambda p(1)(1-p(1))u'(w_1)+\mu(1-p(1))\Delta pu'(w_1) = 0,\]
\[-p(1)(1-p(1))+\lambda p(1)(1-p(1))u'(w_2)-\mu p(1)\Delta pu'(w_2) = 0\]
Subtracting yields

\[ \lambda p(1)(1 - p(1)) (u'(w_1) - u'(w_2)) + \mu \Delta p ((1 - p(1))u'(w_1) + \mu p(1)u'(w_2)) = 0 \]

Since the first term is positive, due to \( u'(w_1) > u'(w_2) \), the second term must be negative.

This requires that \( \mu^* > 0 \)

Because \( \lambda^*, \mu^* > 0 \) the agent’s participation constraint and incentive constraint must bind.
We thus have two equations and two variables $w_1$ and $w_2$

We yield

$$u(w_1^*) = p(1)u(W - s) + (1 - p(1))u(W) + \frac{\psi(1 - p(1))}{\Delta p}$$

$$u(w_2^*) = p(1)u(W - s) + (1 - p(1))u(W) - \frac{p(1)\psi}{\Delta p}$$

Using the inverse of the function $u$ we get

$$w_1^* = u^{-1}\left(p(1)u(W - s) + (1 - p(1))u(W) + \frac{\psi(1 - p(1))}{\Delta p}\right)$$

$$w_2^* = u^{-1}\left(p(1)u(W - s) + (1 - p(1))u(W) - \frac{p(1)\psi}{\Delta p}\right)$$
Exercise 3.10

- Suppose Justin Bieber is giving a concert in Kaiserslautern.
- Justin Bieber can either work, i.e., invest effort $e = 1$, or shirk, i.e., invest $e = 0$.
- His effort costs are $c(e) = \psi e$, with $\psi > 0$.
- His outcome/output can be either high, 2, or low, 1.
- You are the organizer of the concert and want to make sure that he works.
- Unfortunately, neither Justin Bieber’s output (music?) nor his effort is contractible.
Instead, you can contract on some opinion poll which is held among the Beliebers (i.e., his “fans”)

- The outcome of the poll can either be good, $g$, or bad, $b$
- The probability that the poll is good is $p(e) = \frac{1}{4} + \frac{e}{2}$
- Justin Bieber is protected by limited liability, so $w_g, w_b \geq 0$, and his reservation utility is zero

Determine the optimal contract

Solution: $w_g^* = 2\psi$, $w_b^* = 0$